Tests of Homogeneity of Variances of Normal Linear Models

A.B.M. Nur Enayet Talukder

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ABSTRACT

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A.B.M. Nur Enayet Talukder

Many test procedures have been proposed in the literature to detect the heteroscedasticity of statistical populations. Each of the tests has its own assumptions regarding the population distributions. Some of the tests were developed with a view to detect heteroscedasticity in linear models. These tests can also be used to test the equality of variances of normal populations with common mean. Since the classical methods of 'optimum' test constructions did not work well in this situation, researchers developed these tests through some 'ad hoc' approach. In this thesis attempt is made to find a Bayesian solution to test equality of variances of $k(\geq 2)$ normal populations with common location parameters and to extend it for detecting heteroscadasticity in $k(\geq 2)$ linear models with common regression parameters.

In Chapter II of this thesis we consider the Bayesian solution for testing equality of variances of two normal populations with common means. This method is compared with the classical F-test in terms of highest posterior density intervals and is found to have shorter widths. The Bayesian solution is extended for testing equality of variances in two linear models with common regression parameters in Chapter III, the method is applied to a real life problem (Boot and DeWitt's (1960) data) and is observed to give better result than F-test. The extension of

the Bayesian method discussed in Chapter II and III, for the case of more than two normal populations is difficult. Finally, a 'likelihood ratio type' test, proposed by Chaubey (1980) is considered for the detection of heteroscedasticity in $k(\geq 2)$ linear models. This test is studied in detail for a particular case to investigate its properties. A chi-square approximation, which seems to be reasonable is proposed for the test statistic.

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CHAPTER I

INTRODUCTION

1.1 Introduction

Many statistical theories have been developed on the basis of some assumptions on the nature of the random variation present in a set of data or among similar statistical processes, e.g. the theory of linear models, time series analysis, analysis of variance. For an efficient statistical analysis it is necessary to test the assumptions before applying the method of analysis relying on them.

Sometimes it may be of direct interest to test the equality of variances of different populations. Many statistical methods have been developed to test such a hypothesis under specified distributional structure, such as F-test for testing equality of variances of two normal populations, Bartlett's test and Hartley's test for testing equality of variances of k normal populations. All these tests assume that the data have come from normal populations with different location parameters. However, there are situations where data may come from populations with the same location parameter, e.g. in making inference about the precision of two measuring instruments (see Grubbs, 1948). Under such situations how can we test the equality of the variances 'efficiently'? The word 'efficiently' means the optimality of the test. Many researchers have worked on this problem such as Chaubey (1980,1981), Harrison and McCabe (1979), Theil (1971) and Goldfeld and Quandt (1965), but most of these consider a general linear model set-up. Most of the tests are suggested on an ad hoc basis. Some of the tests are very complicated in terms of their distributions and/or their computations while others may not use all the available information.

In this thesis an attempt is made to find tests which use all the information present in the sample data and also have some theoretical basis. With this spirit we investigate a Bayesian solution of the problem. Bayesian solution is difficult for $k(> 2)$ population case. For such a situation we consider the proposal of Chaubey (1980) of a 'likelihood ratio type test' which is computationally simple and uses all the sample data. In the following section (see 1.2) we discuss some preliminaries of the Bayesian Inference and Section 1.3 presents the detailed outline of the thesis.

1.2 Some Preliminaries of Bayesian Inference

We here provide some preliminaries of Bayesian Inference. However for a detailed treatment the reader is referred to the excellent text by Box and Tiao (1973).

In Bayesian inference, the parameter θ about which the inference is to be made, is considered as a random variable with a specified distribution. This distribution is called the <u>prior distribution</u> of θ. Given the prior distribution $p(\theta)$, the corresponding <u>posterior distribution</u> $p(\theta|Y)$ of θ given the observed data (Y) is obtained as

$$p(\theta|Y) \propto L(\theta|Y)\, p(\theta) \tag{1.2.1}$$

where $L(\theta|Y)$ is the likelihood function of θ given the data

$Y = (y_1, y_2, \ldots, y_n)$. The inference about θ is then based on $p(\theta|Y)$. From (1.2.1) it is clear that the selection of prior for θ is important in Bayesian analysis. Some guidelines for the choice of a prior distribution have been put forward by Raiffa and Schlaifer (1961). However, Sir Harold Jeffreys (1961) first gave a rule for the choice of 'non-informative' prior. According to Jeffreys, the prior distribution for a parameter θ is approximately non-informative if it is taken proportional to the square root of Fisher's information measure. He justified this rule on the grounds of its invariance under parameter transformation.

In Bayesian analysis usually interval estimation about the parameter θ is done through the highest posterior density (H.P.D.) intervals of a specified probability content. In the next section we give the definition of H.P.D. interval and an algorithm to construct such an interval.

1.2.1 <u>Highest Posterior Density (HPD) Intervals</u>

<u>Definition</u>: The interval (θ_L, θ_U) is said to be Highest Posterior Density interval of content $(1 - \alpha)$ for θ if

i) $\Pr\{\theta \in (\theta_L, \theta_U)\} > \Pr\{\theta \notin (\theta_L, \theta_U)\}$

ii) $(\theta_U - \theta_L)$ is minimum.

and iii) $\displaystyle\int_{\theta_L}^{\theta_U} p(\theta|Y)\,d\theta = 1 - \alpha$.

The above conditions lead us to minimize $(\theta_U - \theta_L)$ subject to the constraint

$$\int_{\theta_L}^{\theta_U} p(\theta|Y)\,d\theta = 1 - \alpha .$$

To get the required interval we minimize the following function with respect to θ_L, θ_U.

$$G(\theta_L, \theta_U, \lambda) = (\theta_U - \theta_L) + \lambda\left(\int_{\theta_L}^{\theta_U} p(\theta|Y)d\theta - (1-\alpha)\right) \quad (1.2.2)$$

Taking the derivative of (1.2.2) and setting to zero we get,

$$\frac{\partial G}{\partial \theta_U} = 1 + \lambda \frac{\partial}{\partial \theta_U} P(\theta_U|Y) = 0 \ ,$$

and

$$\frac{\partial G}{\partial \theta_L} = -1 - \lambda \frac{\partial}{\partial \theta_L} P(\theta_L|Y) = 0 \ , \quad (1.2.3)$$

where $P(\theta|Y)$ is the posterior distribution function, which imply,

$$p(\theta_U|Y) = p(\theta_L|Y) \ . \quad (1.2.4)$$

Thus, (θ_L, θ_U) is an HPD interval of content $(1-\alpha)$ for θ if

$$\int_{\theta_L}^{\theta_U} p(\theta|Y)d\theta = 1 - \alpha \quad (1.2.5)$$

and

$$p(\theta_L|Y) = p(\theta_U|Y) \ . \quad (1.2.6)$$

The condition (1.2.6) implies that if we have a unimodal posterior pdf we choose the interval around the mode. Choosing the interval (θ_L, θ_U) in this way, satisfies the conditions (i), (ii) and (iii), hence (θ_L, θ_U) satisfying (1.2.5) and (1.2.6) is a HPD interval.

HPD interval is not invariant under parameter transformations. For this reason we consider HPD interval for a function of θ with locally uniform non-informative prior. The interval for θ based on this HPD interval is called the "standardized" HPD interval.

1.2.2 An Algorithm To Find The HPD-Invervals

Let (θ_L, θ_U) be the HPD interval of content $(1-\alpha)$ for θ based on $p(\theta|Y)$. That is (θ_L, θ_U) is such that

$$\int_{\theta_L}^{\theta_U} p(\theta|Y)d\theta = 1 - \alpha$$

and the value of $p(\theta|Y)$ is equal at $\theta = \theta_L$ and $\theta = \theta_U$. However, to get the standardized HPD intervals (in our case) we have to consider the posterior pdf of $u = \log\theta$ given Y. The standardized HPD interval is computed from $(\theta_{LS}, \theta_{US})$ where $(\theta_{LS}, \theta_{US})$ are such that

$$\int_{\ln\theta_{LS}}^{\ln\theta_{US}} p(u|Y)du = 1 - \alpha$$

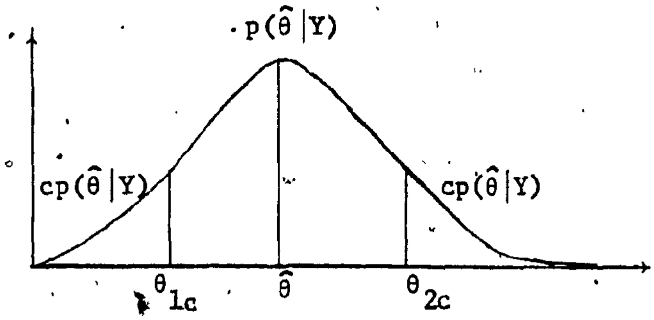
and

$$p(u|Y)\Big|_{u=\log\theta_{LS}} = p(u|Y)\Big|_{u=\log\theta_{US}} .$$

The following iterative procedure can be adopted to find (θ_L, θ_U). Let $\hat{\theta}$ be the mode of $p(\theta|Y)$ then the equation

$$p(\theta|Y) = cp(\theta|Y) ; \qquad 0 < c < 1 \tag{1.2.7}$$

has two roots (Fig.1.2) for every c in the interval $(0,1)$. Let us call them θ_{1c}, θ_{2c}.



(Fig. 1.2)

We vary c over the range $(0, 1)$ and choose that value of c for which the two solutions θ_{1c}, θ_{2c} of the equation (1.2.7) are such that

$$\int_{\theta_{1c}}^{\theta_{2c}} p(\theta|Y)\,d\theta = 1 - \alpha ..$$

The interval $(\theta_{1c}, \theta_{2c})$ is the required HPD interval of content $(1 - \alpha)$ for θ.

1.3 Outline Of The Thesis

As discussed in the beginning of this chapter that our interest is in testing equality of the variances of normal populations, we consider the problem of testing equality of variances of two normal populations with same mean μ, in Chapter II, where some earlier proposed tests are briefly described and a Bayesian solution is proposed for the problem. In Chapter III we consider the problem of testing equality of variances of two populations where the means of the populations depend on some common factors in a linear set up. We describe the tests proposed by Chaubey (1981) and Theil (1971) in this chapter. A Bayesian solution for this case is proposed and compared with F-test. We consider the problem of testing equality of $k(\geq 2)$ linear models in Chapter IV. In this chapter we describe the tests proposed by Goldfeld and Quandt, Harrison and McCabe (1979). A proposal is made to use Bayesian analysis using the method described in Chapter III by grouping the data by some criteria as was done by Goldfeld and Quandt (1965), and Harrison and McCabe (1979). But for $k(> 2)$ populations, a straight forward Bayesian solution is difficult and we propose a 'likelihood ratio type' test for this case. An approximation for the distribution of the test criterion in terms of a chi-square distribution is proposed. A special case is investigated in detail to study the properties of the test.

CHAPTER II

TESTING EQUALITY OF VARIANCES OF TWO NORMAL POPULATIONS WITH EQUAL MEANS

2.1 Introduction

Testing equality of variances of two populations with equal means commonly arises in a situation described as follows. Let μ be the true length of a rod which we wish to measure using two measuring instruments. We want to compare the precision of the instruments i.e. we want to test the hypothesis H_o: $\sigma_1^2 = \sigma_2^2$, where σ_1^2 and σ_2^2 are the variances of the two instruments, using the sample data (see also Grubbs, 1948). The more general situation, where the observations depend on other factors, is a particular case of testing homoscedasticity in a general linear model (see Theil, 1971, pp. 214-218). This general case will be discussed in Chapter III. However, because of the importance of the two sample problem we discuss it in detail in this chapter.

There are situations where we have two independent samples from two normal populations with different means μ_1 and μ_2 respectively, e.g. in analysis of variance, when the experimental error appears to vary considerably from treatment to treatment, so that a preliminary test for the homogeneity of the error variances is required before we proceed to investigate the treatment means. It is well known that the best test of H_o in this situation is provided by the F-test. The Bayesian solution for testing H_o in this case, with the assumption of noninformative reference prior also gives the F-test (see Box and Tiao, 1975, pp. 110). Our focus is mainly on the equal mean situation discussed at the beginning of this chapter. A considerable amount of work has been done in the literature on this problem (see Chaubey, 1981 and reference therein), most of which discuss the problem in a general linear model set-up. These tests can be adapted in the situation discussed here, as a particular case. Some of the tests have been critically reviewed by Chaubey (loc.cit.). In the same paper he proposed a test based on the average of the squared residuals and named it average squared residual test (ASR-test). ASR-test is a modification of the test proposed by C.R. Rao (1970) using the MINQUE of the variances, which in some cases may produce negative estimates for the variances. ASR-test depends on the least squares residuals like the test proposed by C.R. Rao, but it removes the difficulty of using negative estimates of the variances produced by the method of MINQUE.

Some of these tests are described in section 2.2 and a Bayesian solution for the problem is presented in section 2.3. Section 2.4 presents an empirical comparison of the different test procedures.

2.2 Some Tests of H_o

Let $y_{11}, y_{12}, \ldots, y_{1n_1}$ and $y_{21}, y_{22}, \ldots, y_{2n_2}$ be two independent random samples of size n_1 and n_2 from two normal populations with equal mean μ and variances σ_1^2 and σ_2^2 respectively. We wish to test $H_o: \sigma_1^2 = \sigma_2^2$. Below we present some common tests for H_o.

F-Test

The following test statistic is used to test H_o,

$$R_1 = \frac{s_1^2}{s_2^2} , \qquad (2.2.1)$$

where $s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 , \bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$;

$i = 1,2$, $j = 1,2,\ldots,n_i$. The statistic R_1 has the F-distribution with $(n_1 - 1, n_2 - 1)$ degrees of freedom. This test does not take into consideration the equality of the population means.

ASR-Test

ASR-test procedure leads to the following test statistic for the particular problem we are discussing in this chapter,

$$R_2 = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} \qquad (2.2.2)$$

where $\hat{\sigma}_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2$, $\bar{y} = \frac{1}{n} \sum_{i=1}^{2} \sum_{j=1}^{n_i} y_{ij}$ and

$n = n_1 + n_2$. The test statistic R_2 is based on the MINQ estimates of the variances σ_i^2 (see Rao and Chaubey, 1978 for MINQE).

ASR-test is empirically found (Chaubey 1981) to be locally more powerful than the F-test. But the exact distribution of the ASR-test statistic is complicated. Chaubey (1981) also gave an approximation of the distribution for the test statistic R_2.

BLUS Test

The adaptation of Theil's (1971) procedure leads to the following test statistic

$$R_3 = \frac{\sum_{j=1}^{\nu_1} \hat{\varepsilon}_{1j}^2/\nu_1}{\sum_{j=1}^{\nu_2} \hat{\varepsilon}_{2j}^2/\nu_2} \tag{2.2.3}$$

where $\tilde{\varepsilon}_{ij}$ $i = 1,2$ are BLUS (best linear unbiased with scalar covariance) residuals obtained from the least squares residuals obtained from the least squares residuals $e_{ij} = y_{ij} - \bar{y}$ $(i = 1,2\,;\, j = 1,2,\ldots,n_i)$, ν_1 and ν_2 are respective degrees of freedom such that $\nu_1 + \nu_2 = n-1$. Under H_o, R_3 has the F-distribution with (ν_1, ν_2) degrees of freedom. This test is compared in terms of power in Chaubey (1981) and is found to be less powerful than the tests based on the test statistics R_1 and R_2. Though the distribution of R_3, under H_o is simple, its construction is arbitrary and complicated. Goldfeld and Quandt (1965) considered the test statistic R_1 for testing homogeneity of variances by dividing the total sample in two halves.

There is no clear statistical theory behind the derivation of the test statistics discussed so far. In the next section we are presenting a Bayesian method for testing H_o.

2.3 A Bayesian Analysis

The null hypothesis H_o: $\sigma_1^2 = \sigma_2^2$ is equivalent to H_o:$(\sigma_1^2/\sigma_2^2) = 1$. We shall consider Bayesian inference on the ratio $\sigma_1^2/\sigma_2^2 = w$, on the basis of the posterior probability density function of w given the knowledge of the data, Y i.e. $p(w|Y)$. When there is a posterior evidence that the ratio $w = \sigma_1^2/\sigma_2^2$ is close to one, we may accept H_o and if this ratio significantly differs from one, we reject H_o. Inference about w is made by constructing Highest Posterior Density (HPD) intervals of w. First we derive the posterior distribution of w.

2.3.1 Posterior Distribution of w

The likelihood function of the observations $Y = \{y_{ij}, i=1,2 ; j=1,2,\ldots,n_i\}$ is

$$L(\mu,\sigma_1^2,\sigma_2^2|Y) \alpha (\sigma_1^2)^{-\frac{n_1}{2}} (\sigma_2^2)^{-\frac{n_2}{2}} \exp\{-\tfrac{1}{2}(\sum_{j=1}^{n_1}\frac{(y_{1j}-\mu)^2}{\sigma_1^2} + \sum_{j=1}^{n_2}\frac{(y_{2j}-\mu)^2}{\sigma_2^2}\}$$

$$; -\infty < \mu < \infty , 0 < \sigma_1^2 < \infty , 0 < \sigma_2^2 < \infty . \qquad (2.3.1)$$

We consider the non-informative reference prior on $(\mu , \sigma_1^2 , \sigma_2^2)$ as

$$p(\mu , \sigma_1^2 , \sigma_2^2) \ \alpha \frac{1}{\sigma_1^2\sigma_2^2} . \qquad (2.3.2)$$

The joint posterior pdf of $(\mu , \sigma_1^2 , \sigma_2^2)$ given Y with respect to the above prior is given by

$$p(\mu,\sigma_1^2,\sigma_2^2|Y) \alpha \ L(\mu,\sigma_1^2,\sigma_2^2|Y)\, p(\mu,\sigma_1^2,\sigma_2^2)$$

$$\alpha \ (\sigma_1^2)^{-\frac{n_1}{2}-1} (\sigma_2^2)^{-\frac{n_2}{2}-1} \exp\{-\tfrac{1}{2}(\sum_{j=1}^{n_1}\frac{(y_{1j}-\mu)^2}{\sigma_1^2} + \sum_{j=1}^{n_2}\frac{(y_{2j}-\mu)^2}{\sigma_2^2})\}$$

$$\alpha \ (\sigma_1^2)^{-\frac{n_1}{2}-1} (\sigma_2^2)^{-\frac{n_2}{2}-1} \exp\{-\tfrac{1}{2}\sigma_1^2(\nu_1 s_1^2 + w\nu_2 s_2^2)\}$$

$$\times \ \exp\{-\frac{n_1}{2\sigma_1^2}((\bar{y}_1-\mu)^2 + \lambda(\bar{y}_2-\mu)^2)\} , \qquad (2.3.3)$$

where

$$s_i^2 = \frac{1}{\nu_i}\sum_{j}^{n_i}(y_{ij}-\bar{y}_i)^2 ,$$

$$\bar{y}_i = \sum_{j=1}^{n_i} y_{ij}/n_i ,$$

$$\nu_i = (n_i - 1) ,$$

$$w = \sigma_1^2/\sigma_2^2 \quad ,$$

and
$$\lambda = \frac{wn_2}{n_1} \quad .$$

Writing

$$(\bar{y}_1 - \mu)^2 + \lambda(\bar{y}_2 - \mu)^2 = (\bar{y}_1 - \bar{y}_2)^2 \frac{\lambda}{1+\lambda} + (\hat{\mu} - \mu)^2 (1+\lambda)$$

where
$$\hat{\mu} = \frac{\bar{y}_1 + \lambda\bar{y}_2}{1+\lambda} \quad ,$$

the posterior pdf of $(\mu, \sigma_1^2, \sigma_2^2)$ given Y becomes

$$p(\mu,\sigma_1^2,\sigma_2^2|Y) \propto (\sigma_1^2)^{-\frac{n_1}{2}-1} (\sigma_2^2)^{-\frac{n_2}{2}-1} \exp\{-\frac{1}{2\sigma_1^2}(\nu_1 s_1^2 + \nu_2 w s_2^2 + \frac{n_1\lambda}{(1+\lambda)}(\bar{y}_1 - \bar{y}_2)^2)\}$$
$$\times \exp\{-\frac{n_1(1+\lambda)}{2\sigma_1^2}(\mu - \hat{\mu})^2\} \quad . \qquad (2.3.4)$$

The marginal posterior pdf of (σ_1^2, σ_2^2) is obtained by integrating out μ from (2.3.4) over its range $-\infty < \mu < \infty$. Thus

$$p(\sigma_1^2, \sigma_2^2|Y) \propto \int_{-\infty}^{\infty} p(\mu, \sigma_1^2, \sigma_2^2 | Y)\, d\mu$$

$$\propto (\sigma_1^2)^{-(\frac{n_1}{2}+1)} (\sigma_2^2)^{-(\frac{n_2}{2}+1)} \frac{\sigma_1}{(1+\lambda)^{\frac{1}{2}}}$$
$$\times \exp\{-\frac{1}{2\sigma_1^2}(s^2(w))\} \qquad (2.3.5)$$

where $s^2(w) = [\nu_1 s_1^2 + w\nu_2 s_2^2 + \frac{n_1\lambda}{1+\lambda}(\bar{y}_1 - \bar{y}_2)^2]$.

Making the following transformation

$$(\sigma_1^2, \sigma_2^2) \rightarrow (\sigma_1^2, \; w = \sigma_1^2/\sigma_2^2)$$

in (2.3.5) and noting that the Jacobian of the transformation is σ_1^2/w^2 , we get the posterior pdf of (σ_1^2, w) as

$$p(\sigma_1^2, w|Y) \propto (\sigma_1^2)^{-\frac{(n_1+n_2+1)}{2}} (1+\lambda)^{-\frac{1}{2}} \exp\{-\frac{1}{2\sigma_1^2}(s^2(w))\} \qquad (2.3.6)$$

$$; 0 < w < \infty , \quad \sigma_1^2 > 0 .$$

Hence the marginal posterior pdf of w is

$$p(w|Y) \propto \int_0^\infty p(\sigma_1^2, w|Y)\, d\sigma_1^2$$

$$\propto (1+\lambda)^{-\frac{1}{2}} w^{\frac{n_2}{2}-1} [s^2(w)]^{-\frac{n_1+n_2-1}{2}} \quad ; \quad 0 < w < \infty . \qquad (2.3.7)$$

The final form of the posterior pdf of w given Y is

$$p(w|Y) \propto (1+\frac{n_2 w}{n_1})^{-\frac{1}{2}} w^{\frac{n_2}{2}-1} [\nu_1 s_1^2 + w\nu_2 s_2^2 + \frac{n_1 n_2 w}{n_1+n_2 w}(\bar{y}_1 - \bar{y}_2)^2]^{-\frac{(n_1+n_2-1)}{2}}$$

$$; 0 < w < \infty . \qquad (2.3.8)$$

Therefore,

$$p(w|Y) = K \cdot g(w) , \qquad (2.3.9)$$

where

$$g(w) = (1+\frac{n_2 w}{n_1})^{-\frac{1}{2}} w^{\frac{n_2}{2}-1} [s^2(w)]^{-\frac{(n_1+n_2-1)}{2}} ,$$

and

$$K^{-1} = \int_0^\infty g(w)\, dw . \qquad (2.3.10)$$

2.3.2 A Transformation for the Computation of K^{-1} and the Distribution of w

The following transformation is used to replace the improper integral in (2.3.10) which helps to get rid of the truncation error involved in a numerical integration of an improper integral.

The transformation used is

$$z = \frac{w}{1+w} \quad ; \quad 0 < z < 1 \quad ,$$

which implies that $w = z/(1-z)$ and $dw = 1/(1-z)^2\, dz$. Hence K in (2.3.10) can be written as

$$K^{-1} = \int_0^1 [g(w)]_{w=z/(1-z)} (1-z)^{-2}\, dz \quad . \tag{2.3.11}$$

The integral in (2.3.11) may be conveniently evaluated using IMSL subroutine DCADRE or any other comparable subroutine.

The posterior pdf of w and $\log w$ are obtained through the distribution of z and those distributions are used to find HPD intervals and standardized HPD intervals for w respectively.

The posterior pdf of w given Y is found using the posterior distribution of z as follows:

$$p(w|Y) = [p(z|Y) \cdot (1-z)^2]_{z=\frac{w}{1+w}} \quad , \tag{2.3.12}$$

and the posterior distribution of $u = \log w$ given Y is obtained as

$$p(u|Y) = [p(z|Y) \cdot z(1-z)]_{z=e^u/(1+e^u)} \quad . \tag{2.3.13}$$

The modes of $p(w|Y)$ and $p(u = \log w|Y)$ are obtained iteratively (or otherwise) by solving respectively, $h(w) = 0$ and $h^*(w) = 0$, where

$$h(w) = (\frac{n_2}{2} - 1)\, w^{-1} - \frac{n_2}{2(n_1+n_2 w)} - \frac{(n_1+n_2-1)}{2s^2(w)}[\nu_2 s_2^2 + \frac{n_1^2 n_2 (\bar{y}_1 - \bar{y}_2)^2}{(n_1+n_2 w)^2}]$$

and

$$h^*(w) = \frac{n_2}{2w} - \frac{n_2}{2(n_1+n_2 w)} - \frac{(n_1+n_2-1)}{2s^2(w)}[\nu_2 s_2^2 + \frac{n_1^2 n_2 (\bar{y}_1 - \bar{y}_2)^2}{(n_1+n_2 w)^2}] \quad .$$

FORTRAN source listing for this purpose is provided in Appendix A2.1 (Subroutine SOLMOD and SOLMOD1) .

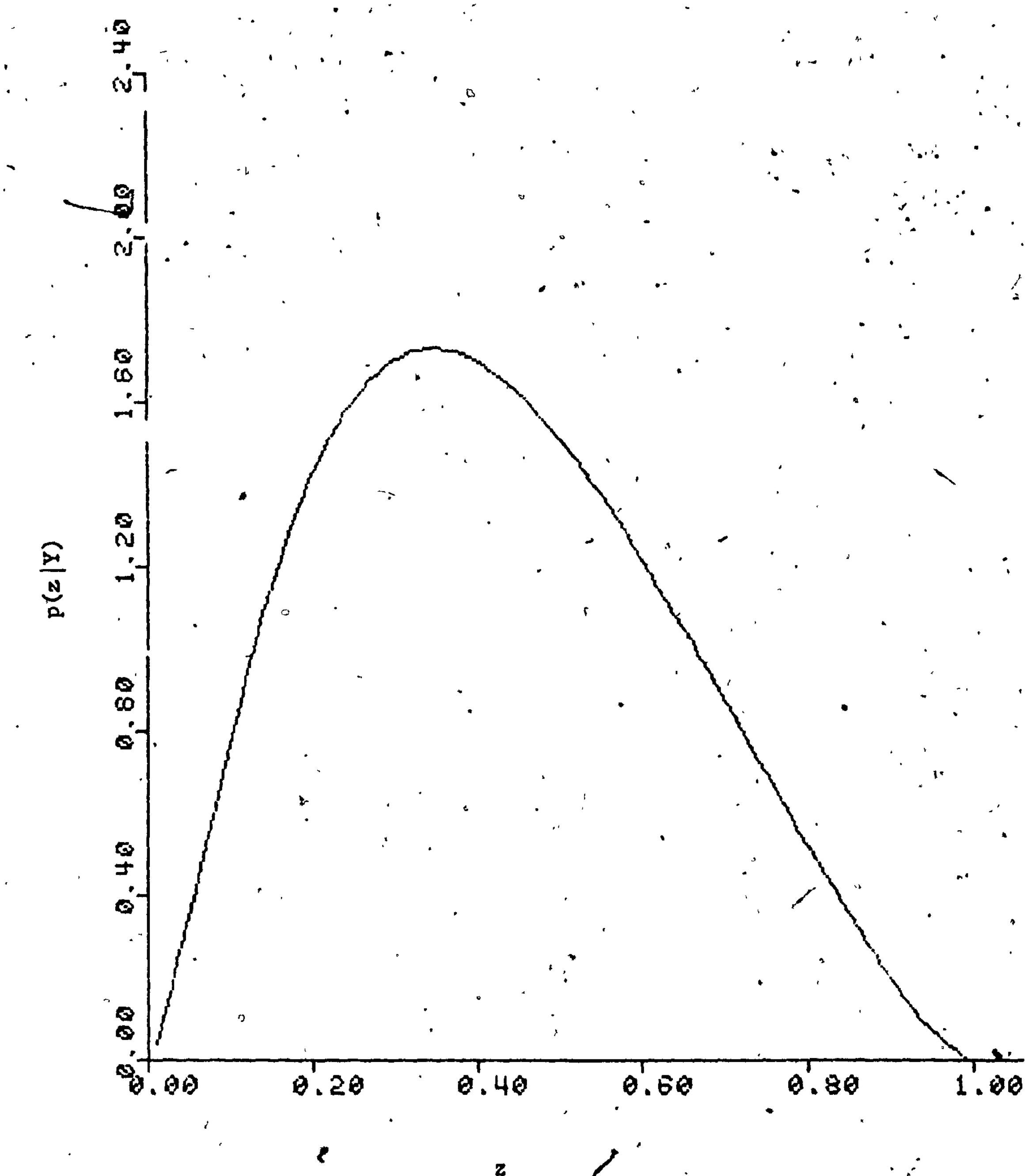


Figure 2.1a POSTERIOR DISTRIBUTION OF z WHEN $n_1 = 5$, $n_2 = 10$, $s_1^2 = .67$, $s_2^2 = 1.03$, $D = .31$.

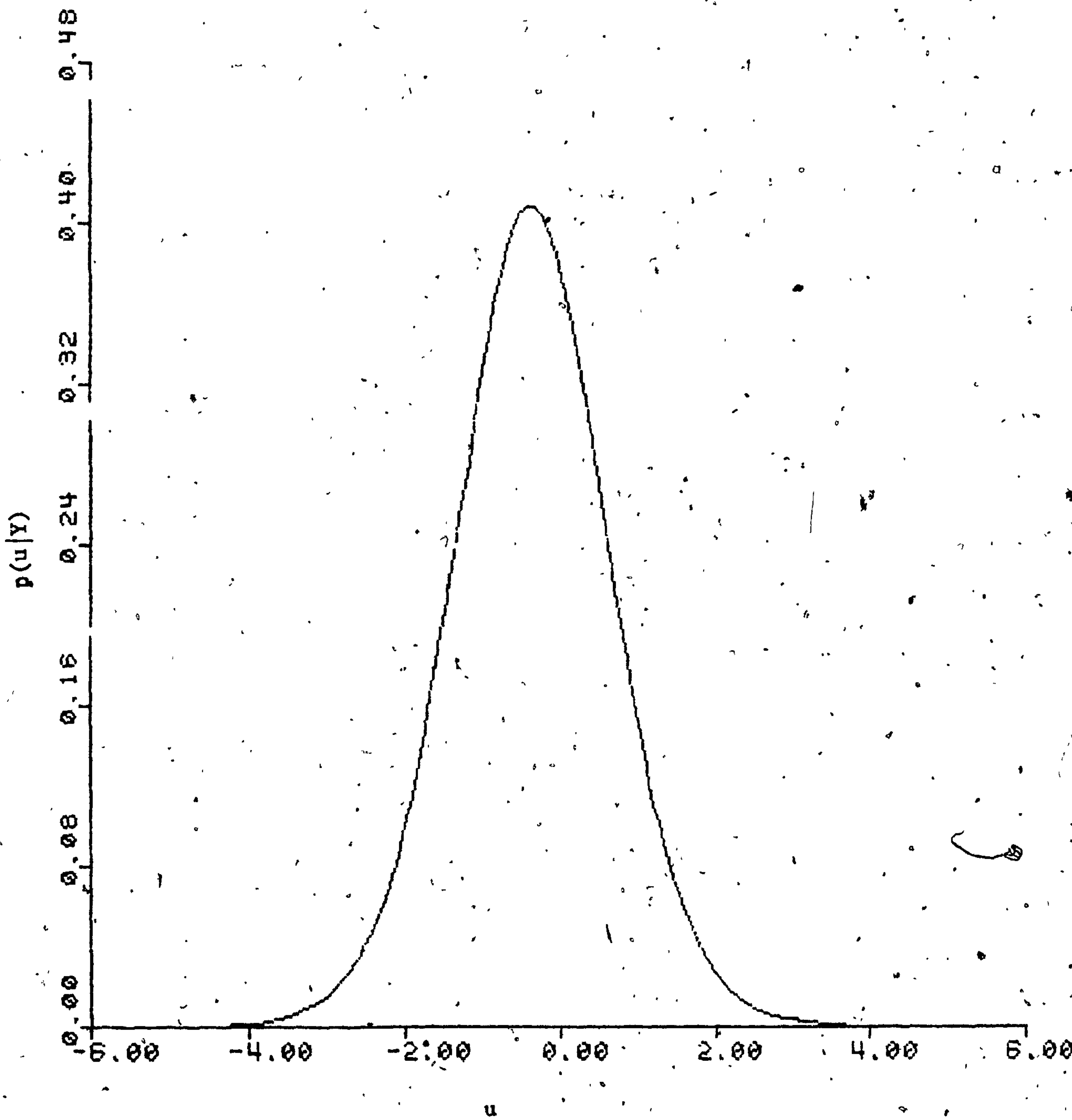


Figure 2.1b POSTERIOR DISTRIBUTION OF u WHEN $n_1 = 5$, $n_2 = 10$,

$s_1^2 = .67$, $s_2^2 = 1.03$, $D = .31$

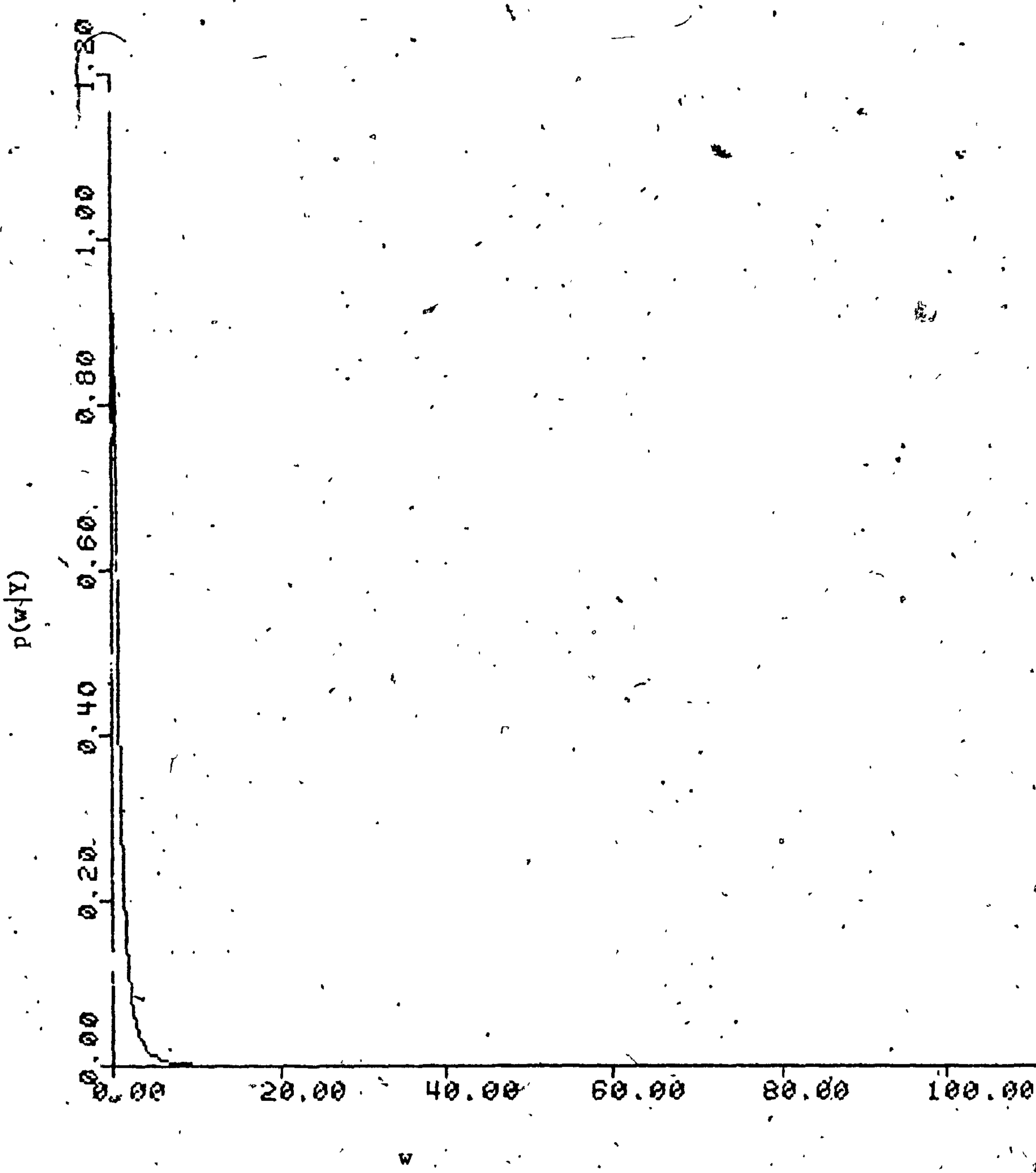


Figure 2.1c POSTERIOR DISTRIBUTION OF w WHEN $n_1 = 5$, $n_2 = 10$,

$s_1^2 = .67$, $s_2^2 = 1.03$, $D = .31$

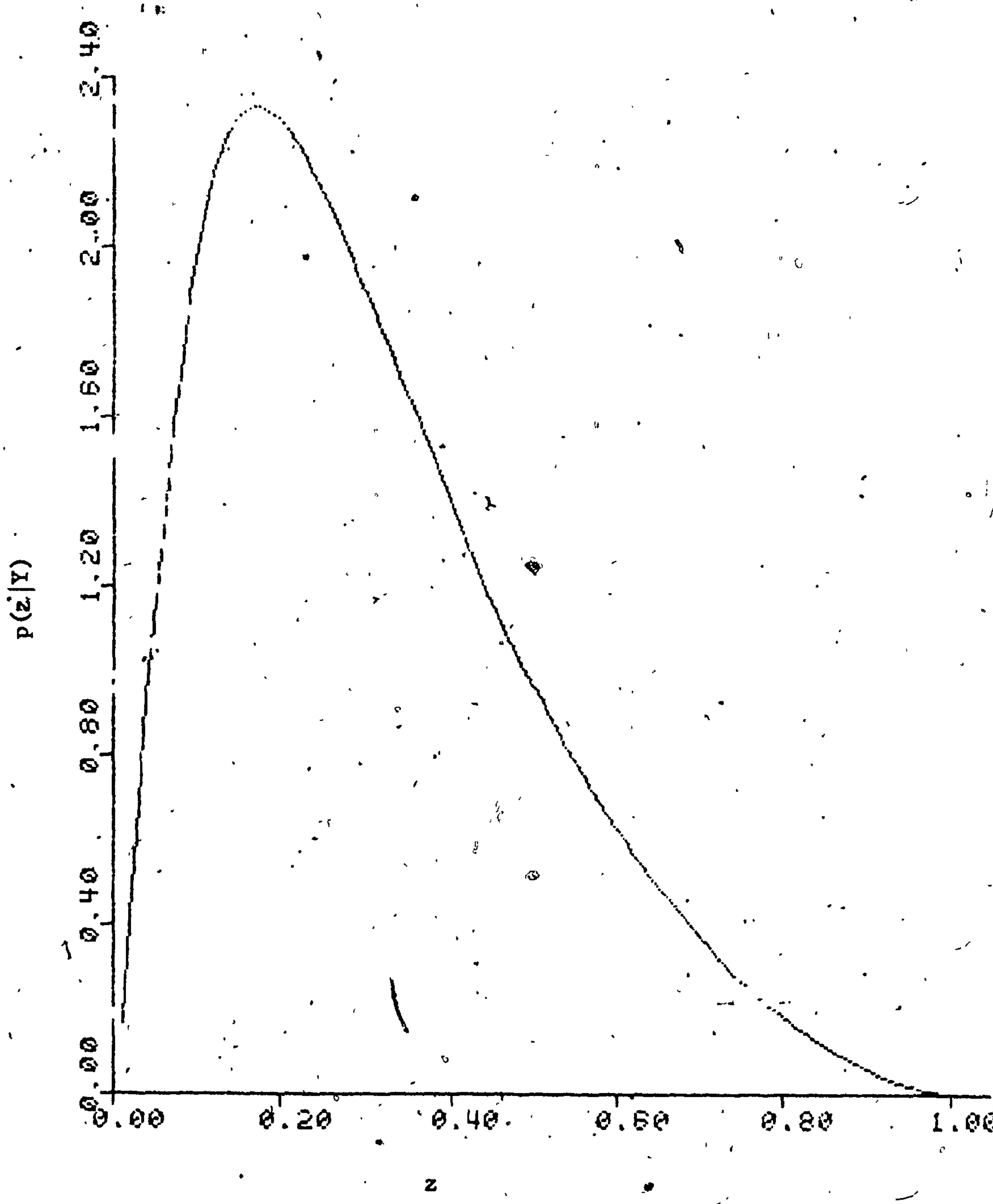


Figure 2.2a POSTERIOR DISTRIBUTION OF z WHEN $n_1 = 5$, $n_2 = 5$, $s_1^2 = 1.71$, $s_2^2 = 4.79$, and $D = -.81$.

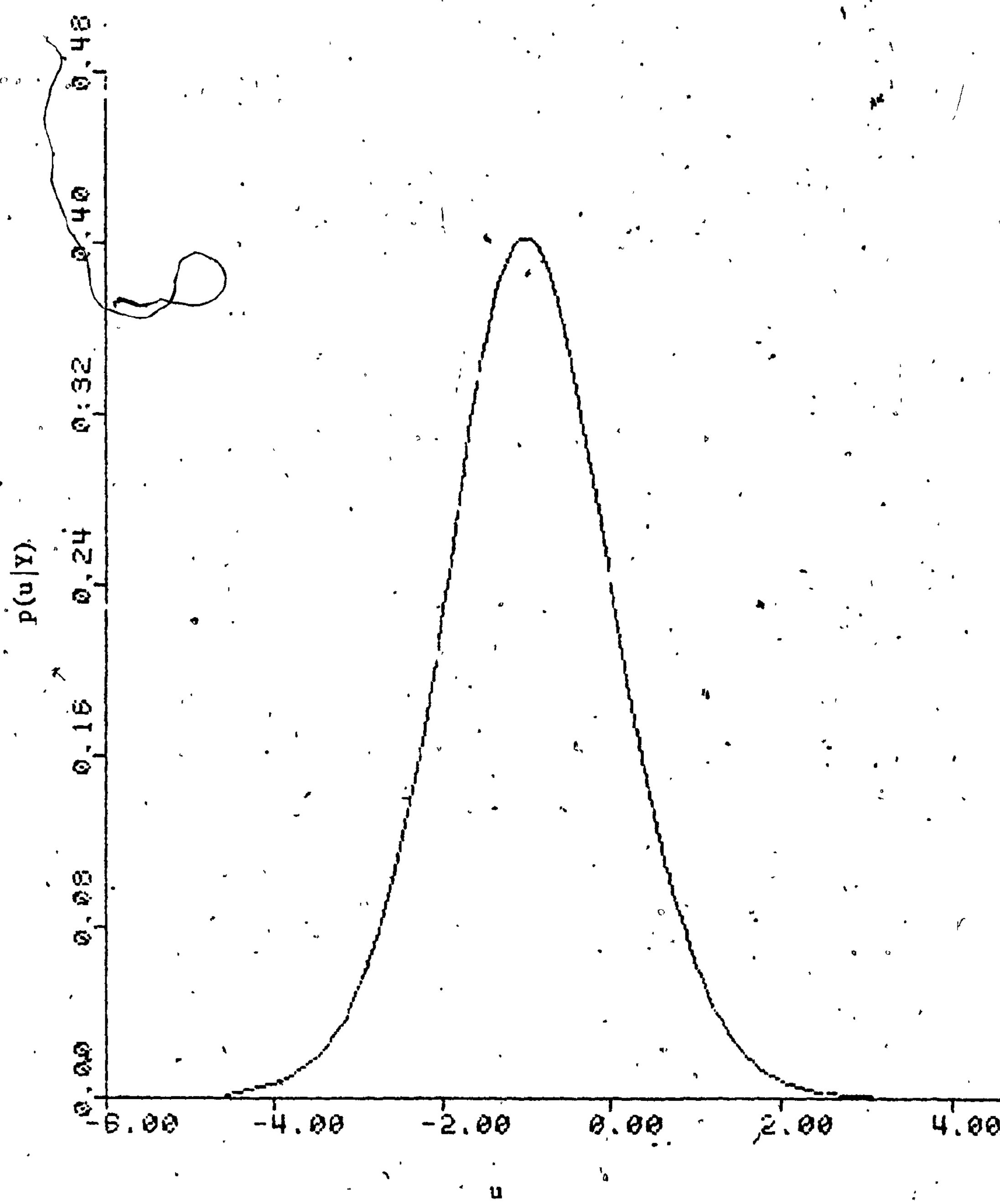


Figure 2.2b POSTERIOR DISTRIBUTION OF u WHEN $n_1 = 5$, $n_2 = 5$,

$s_1^2 = 1.71$, $s_2^2 = 4.79$ and $D = -.81$

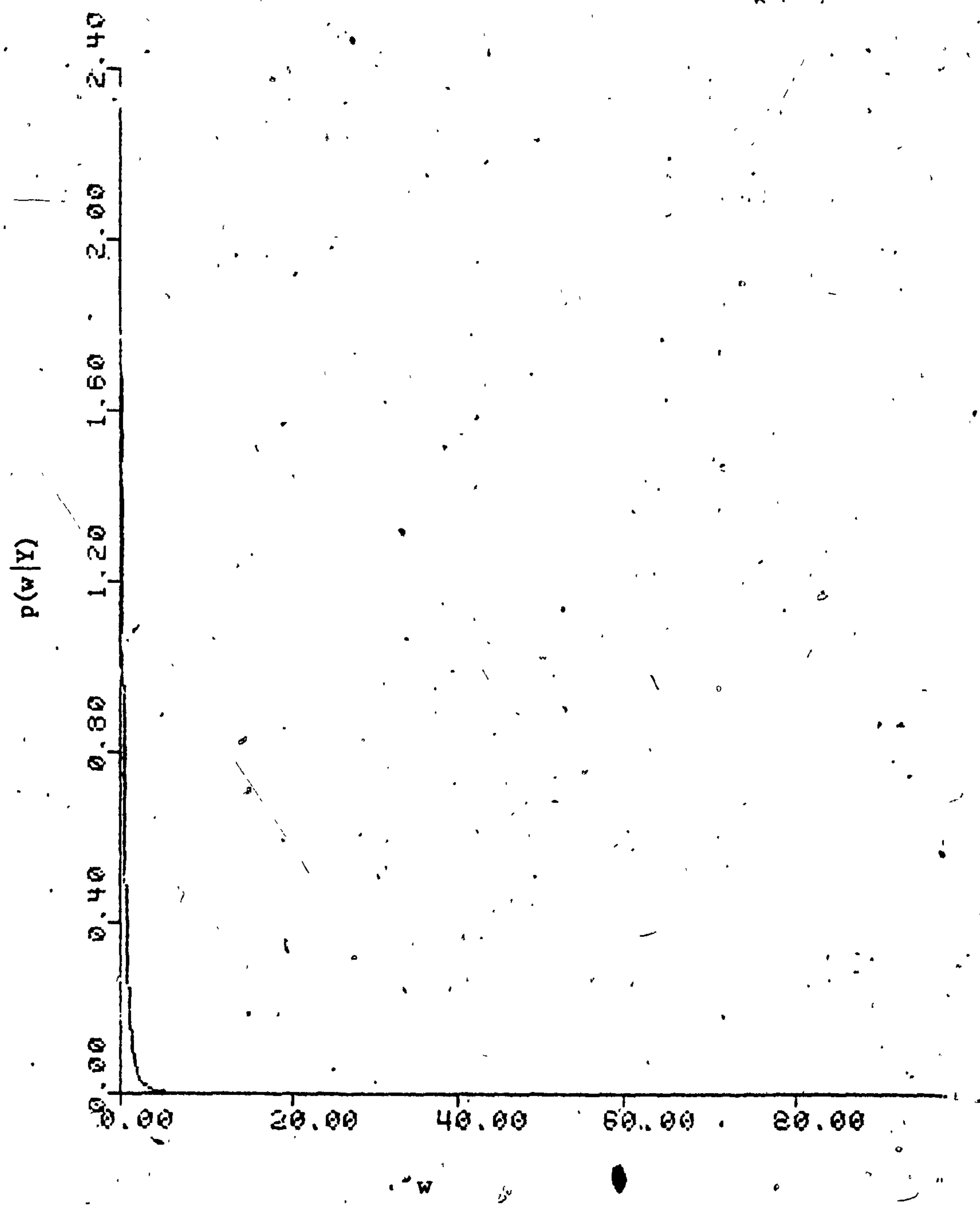


Figure 2.2c POSTERIOR DISTRIBUTION OF w WHEN $n_1 = 5$, $n_2 = 5$, $s_1^2 = 1.71$, $s_2^2 = 4.79$ and $D = -.81$

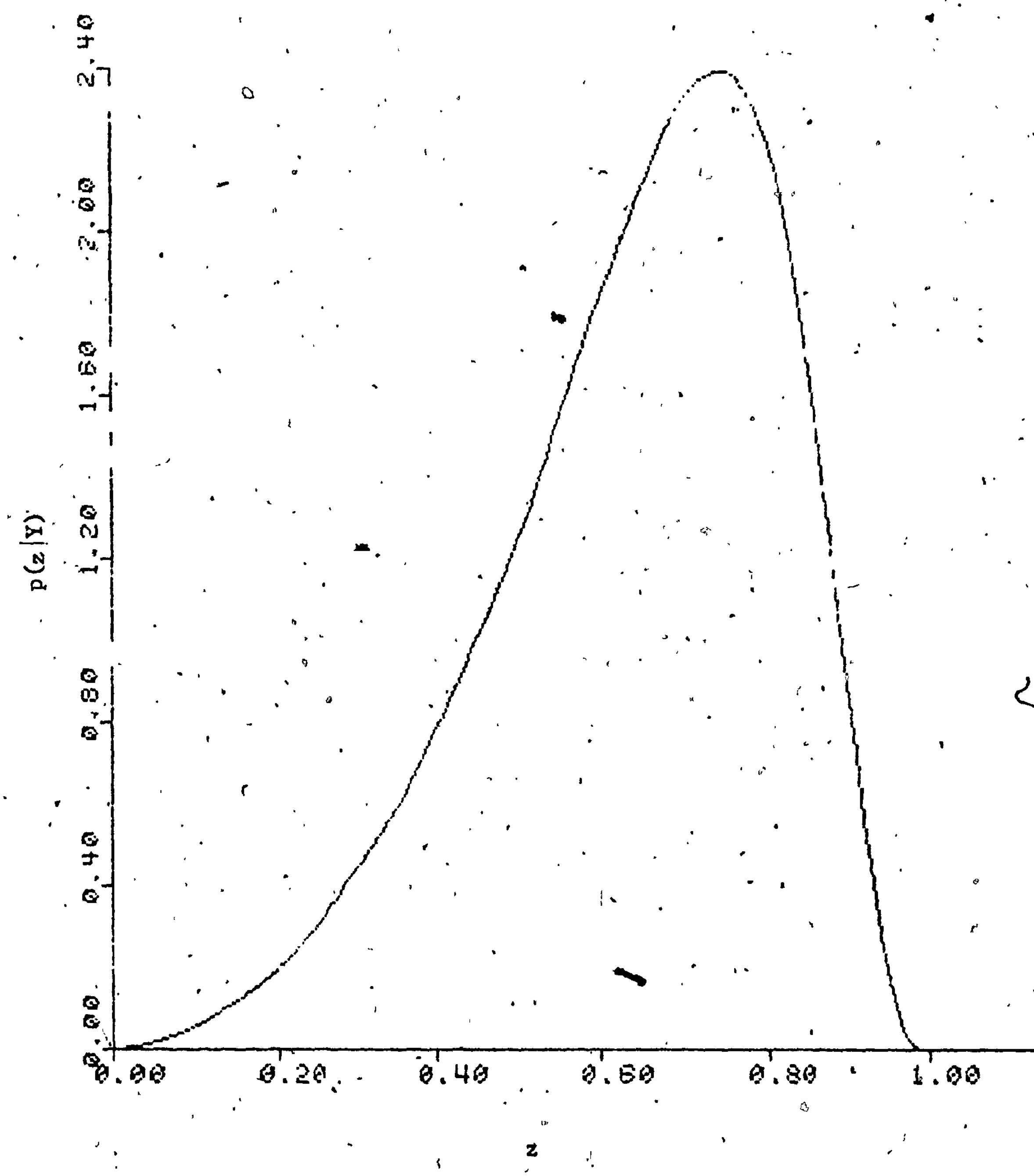


Figure 2.3a POSTERIOR DISTRIBUTION OF z WHEN $n_1 = 10$, $n_2 = 5$,

$s_1^2 = 2.25$, $s_2^2 = 1.10$ and $D = .15$

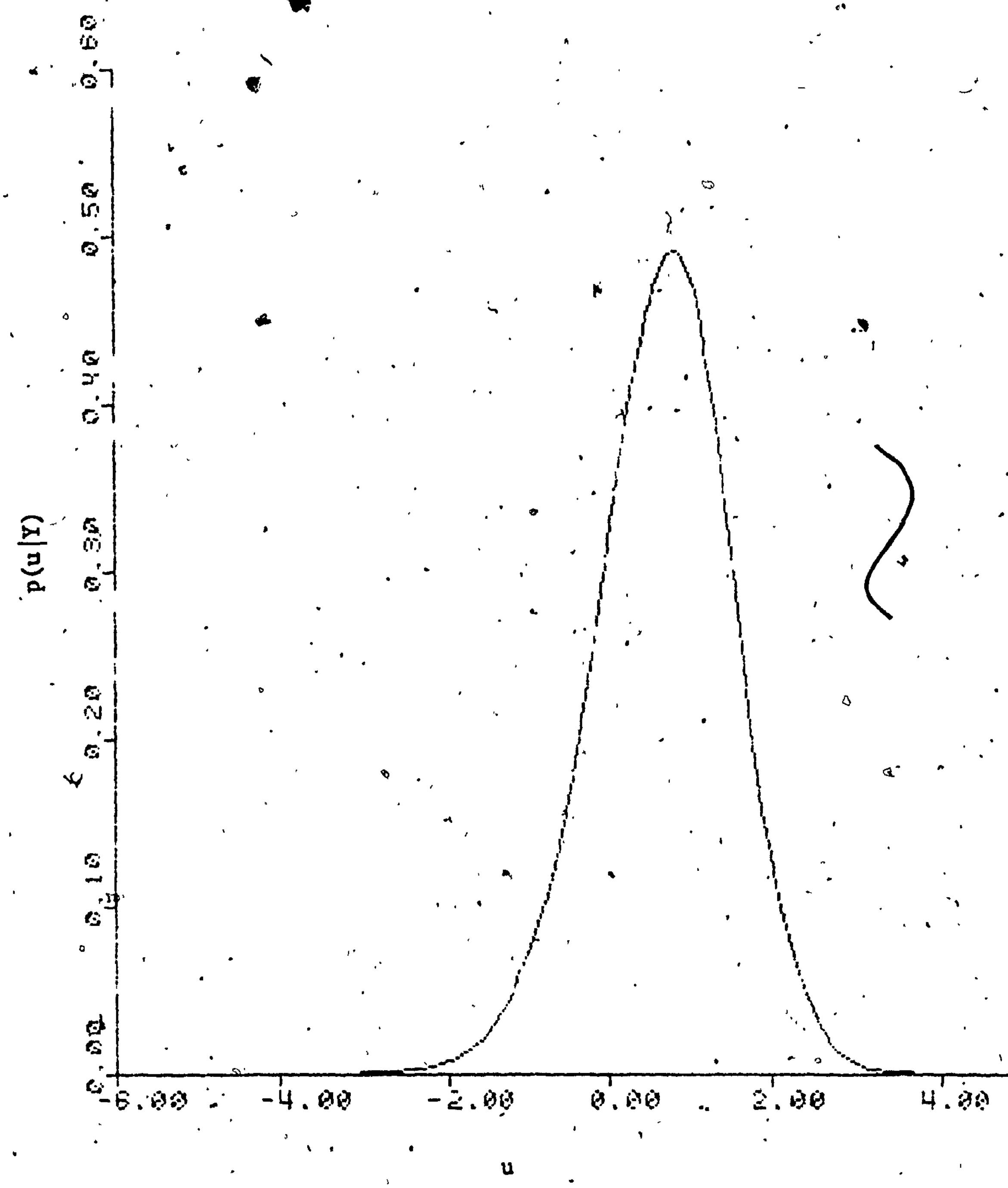


Figure 2.3b POSTERIOR DISTRIBUTION OF u WHEN $n_1 = 10$, $n_2 = 5$, $s_1^2 = 2.25$, $s_2^2 = 1.10$ and $D = .15$

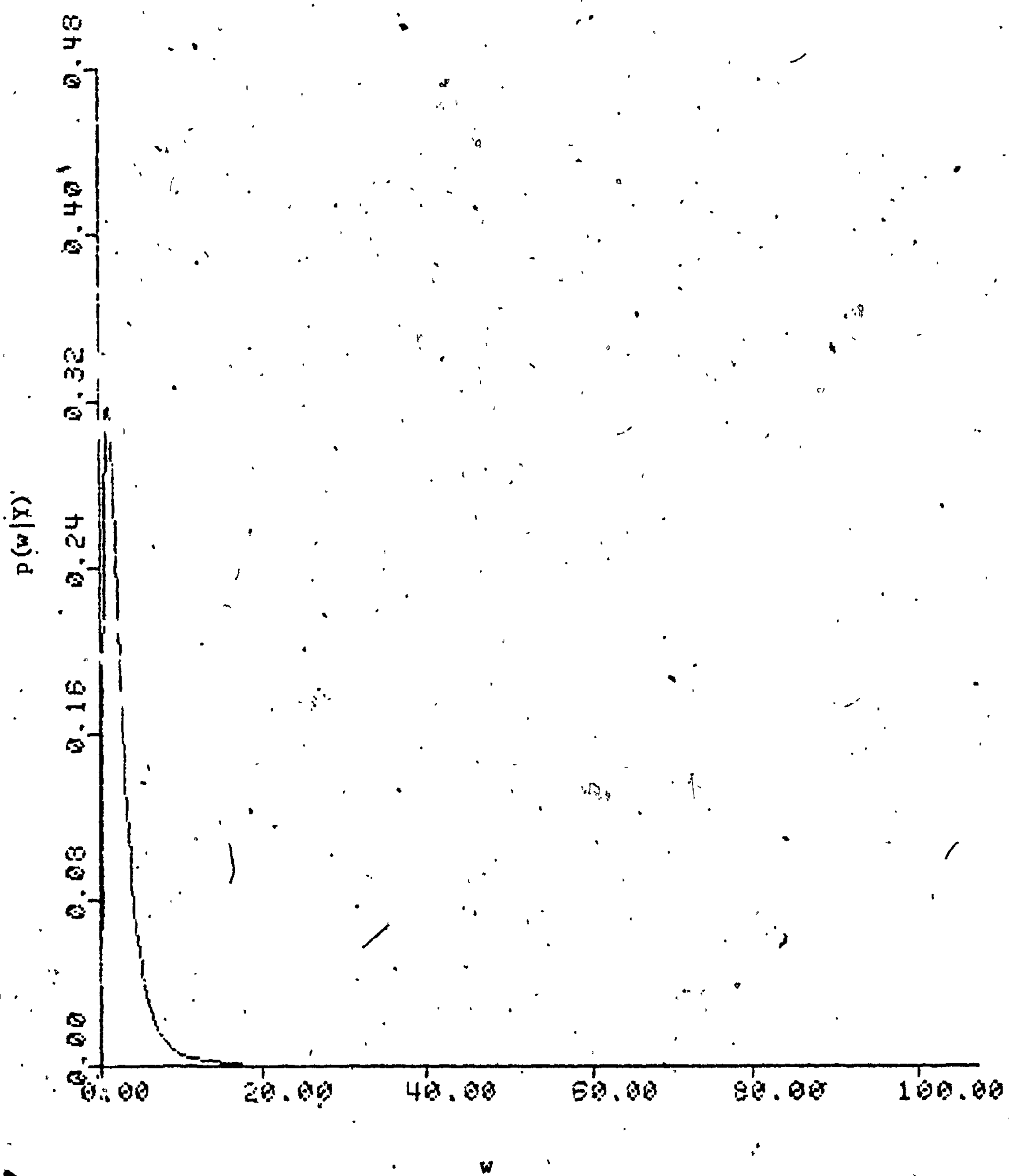


Figure 2.3c. POSTERIOR DISTRIBUTION OF w WHEN $n_1 = 10$, $n_2 = 5$, $s_1^2 = 2.25$, $s_2^2 = 1.10$ and $D = .15$

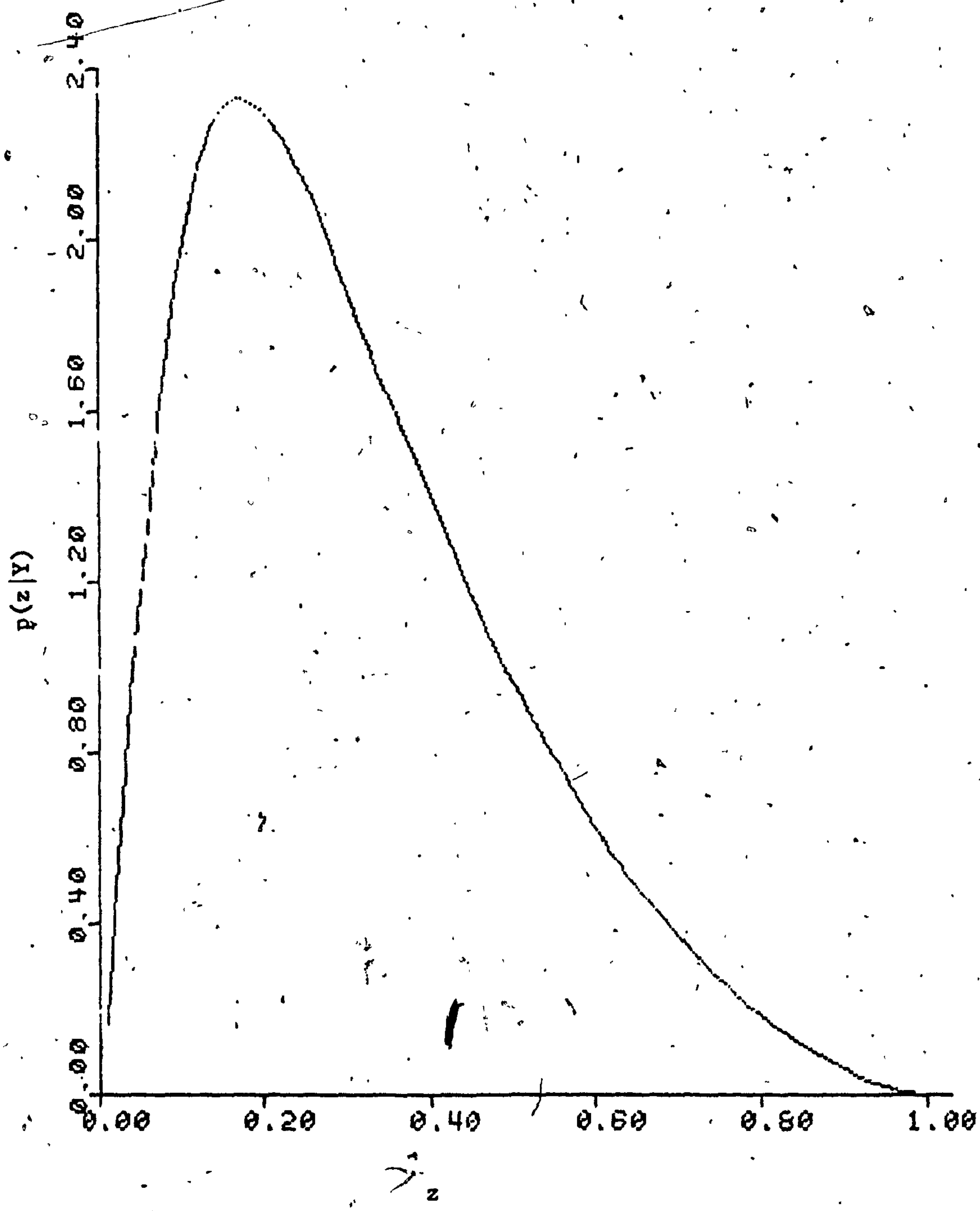


Figure 2.4a POSTERIOR DISTRIBUTION OF z WHEN $n_1 = 5$, $n_2 = 5$,

$s_1^2 = .87$, $s_2^2 = 14.59$, $D = -.65$

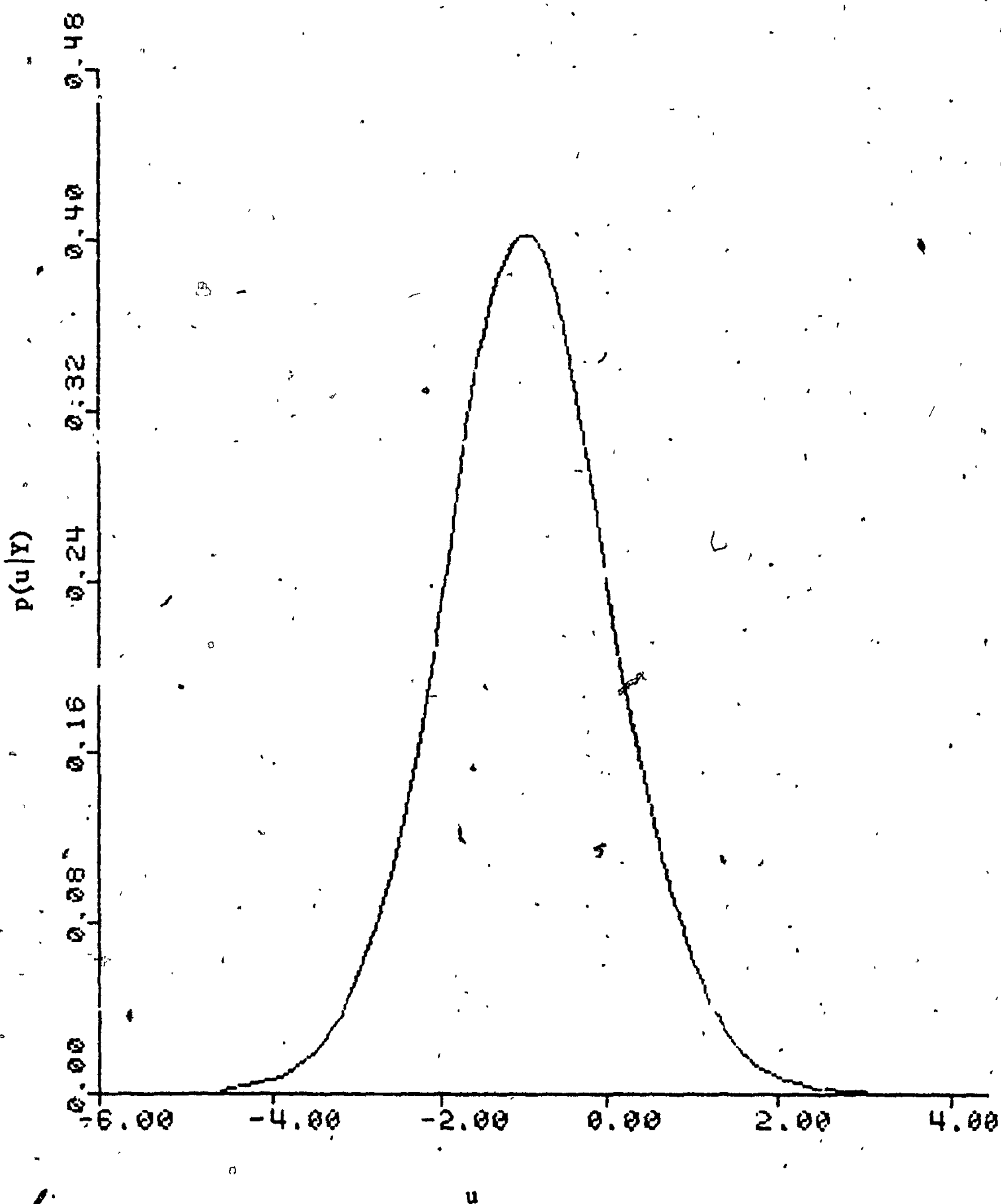


Figure 2.4b POSTERIOR DISTRIBUTION OF u WHEN $n_1 = 5$, $n_2 = 5$, $s_1^2 = .87$, $s_2^2 = 14.59$, $D = -.65$

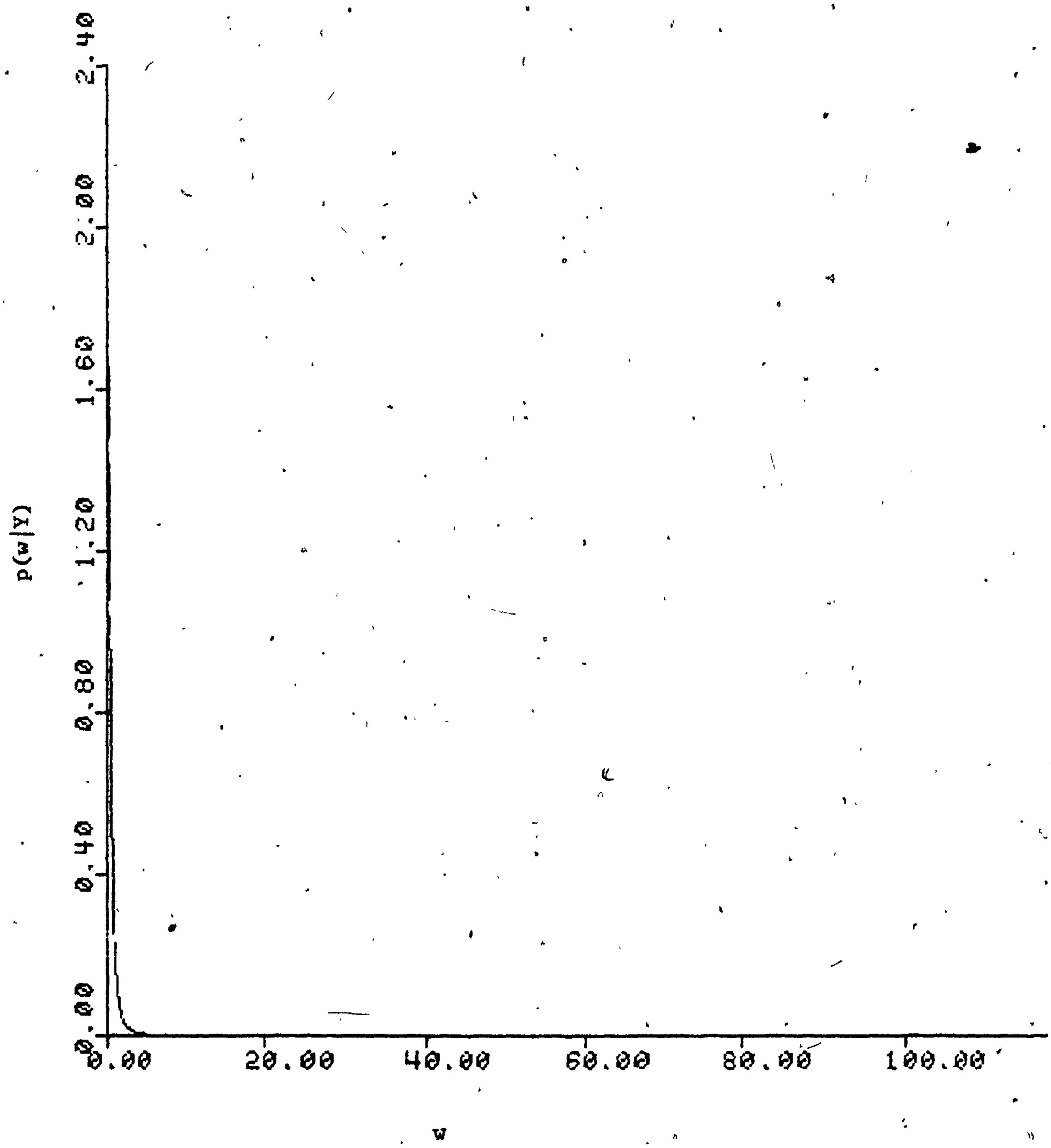


Figure 2.4c POSTERIOR DISTRIBUTION OF w WHEN $n_1 = 5$, $n_2 = 5$,

$s_1^2 = .87$, $s_2^2 = 14.59$, $D = -.65$

The posterior distributions $p(z|Y)$, $p(w|Y)$ and $p(u|Y)$ are computed using the IMSL integrating subroutine DCADRE and are plotted for different values of n_1, n_2, s_1, s_2 and $D = \bar{y}_1 - \bar{y}_2$ (see Figs. 2.1 to 2.4). The fortran program listing is provided in Appendix A2.1. The values of s_1, s_2 and D were randomly generated using GGCHS and GGNQF subroutines of IMSL for generating chi-square and normal random variables. The distribution $p(u|Y)$ looks symmetric in each case, hence inferences based on $\log w$ may be quite appropriate. In the next section we present an empirical comparison of Bayes' procedures based on w and u and the F-statistic.

2.4 An Empirical Comparison of Bayes-Procedure with Other Analysis

We generate s_1^2, s_2^2 and D such that $(n_1 - 1)s_1^2 \sim \chi^2_{n_1-1}$, $(n_2 - 1)ws_2^2 \sim \chi^2_{n_2-1}$ and $D \sim N(0, \frac{1}{n_1} + \frac{1}{wn_2})$ for given n_1, n_2 and w, 100 times. We confine our attention to the small samples because of the computational limitations. (Even for values of (n_1, n_2) as small as (20, 20), computer time needed to generate two samples, compute HPD and standardized HPD intervals 100 times was more than 3000 CPU seconds). On the basis of those generated samples we computed the HPD intervals of content .95, .99 based on $p(w|Y)$ and standardized HPD intervals which are based on $p(\log w|Y)$ using the algorithm described in Chapter I. HPD intervals are also calculated using F-distribution (see Box and Tiao, 1973, Eq. 2.6.2). We compare these intervals in terms of the average number of times they contain the specified ratio w (coverage frequency), which is presented in table 2.1. We did not compare BLUS test since it is found less powerful than F-test (Chaubey, 1981). Also, we could not compare ASR because

TABLE 2.1 COMPARISON OF THE BAYES-PROCEDURE WITH CLASSICAL F-TEST

n_1	n_2	w	COVERAGE FREQUENCY BASED ON w	log w	F	AVERAGE WIDTHS BASED ON w	log w	F
				α = .01				
5	5	.10	98	98	97	3.34	4.61	2.97
		1.0	98	99	97	24.66	34.16	39.68
		2.0	98	99	97	46.95	65.39	59.37
10	10	.10	99	98	98	.69	.82	.74
		1.0	99	98	98	6.63	7.92	7.36
		2.0	100	99	95	24.95	32.93	22.67
5	10	.10	99	99	95	1.72	2.13	.94
		1.0	99	98	95	14.14	17.59	9.43
		2.0	99	99	95	27.59	34.47	18.87
10	5	.10	100	99	95	1.23	1.62	2.42
		1.0	100	99	95	12.89	16.96	24.21
		2.0	100	99	95	25.02	33.03	48.42
				α = .05				
5	5	.10	91	90	84	1.39	2.00	1.40
		1.0	89	88	84	11.11	16.08	13.98
		2.0	88	90	84	21.26	30.94	27.97
10	10	.10	97	97	98	0.40	0.48	0.45
		1.0	97	98	98	3.95	4.79	4.47
		2.0	98	98	98	7.78	9.45	8.94
5	5	.10	93	96	90	0.76	0.98	0.58
		1.0	92	94	90	6.55	8.38	5.77
		2.0	92	94	90	12.82	16.45	11.55
10	5	.10	97	98	89	0.68	0.91	1.16
		1.0	98	97	89	7.37	10.04	11.55
		2.0	98	98	89	14.40	19.73	23.10

the procedure for finding the confidence intervals using the distribution of ASR-test statistic under H_o is not clear. The source listing of the FORTRAN program which generates the samples and finds the HPD intervals mentioned above is given in the Appendix A2.2. This program uses IMSL subroutine ZFALSE and DCADRE to compute HPD intervals. It is observed that HPD-interval based on $p(\log w|Y)$ is almost always better than HPD-interval based on F-distribution on comparing the coverage frequency. The HPD-intervals based on $p(w|Y)$ and F-statistic may have smaller widths but in such cases they loose in terms of coverage frequency.

CHAPTER III

TESTING EQUALITY OF VARIANCES OF TWO LINEAR MODELS WITH COMMON REGRESSION PARAMETERS

3.1 Introduction

It is quite natural in practice that the observations from two sources depend on some common as well as some uncommon factors. But we are treating the case where the observations from two sources linearly depend only on common factors. For example, in analyzing an investment time series data relating to two corporations Boot and DeWitt (1960) assumed that the annual investment is a linear function of expected profitability and real captial at the beginning of the year (see Box and Tiao, 1973, Chapter 9). In this example the data from two sources i.e. the corporations, depend only on two common factors. There are situations where data may depend on $p(>2)$ common factors. Such a situation is represented by the model,

$$Y_i = X_i\beta + \varepsilon_i \; ; \qquad i = 1,2 \tag{3.1.1}$$

where Y_i is an $(n_i \times 1)$ observation vector, X_i is an $(n_i \times p)$ known matrix (matrix of observations on the p factors), β is $(p \times 1)$ unknown parameter vector and ε_i is the $(n_i \times 1)$ disturbance vector, which is assumed to be normally distributed with zero mean and dispersion matrix $\sigma_i^2 I_{n_i}$, ε_1 being independent of ε_2.

Because of the independence of the two sources it is quite natural to expect that the variances in two sources will be different. If they, for some reason, could be assumed to be equal then the data from the two different sources could be pooled under one linear model and the analysis of a single linear model would be applied. Hence to detect the differences in the variances associated with the two linear models is of interest. Thus, we are interested in testing the hypothesis $H_o: \sigma_1^2 = \sigma_2^2$ or equivalently $H_o: \sigma_1^2 / \sigma_2^2 = 1$. This is a general case of the problem discussed in Chapter II. As pointed out before, because of the generality of this problem, it has attracted the attention of many researchers.

Goldfeld and Quandt (1968) considered the problem of testing the homogeneity of variances of a linear model against the alternative that the variances are not equal. They divided the observations into two groups by means of some criterion and then computed the sum of squares of residuals from the regressions fitted seperately to the two groups by least squares method and used the ratio of resultant residual sum of squares as a test statistic. In this approach, the test statistic has F-distribution under the null hypothesis, even if the regression parameters of two linear models are different. They have also proposed a non-parametric test for testing H_o.

We already mentioned in Chapter II that Theil (1971, section 5.2) proposed a test for testing H_o using what he called BLUS (Best linear unbiased with scalar covariance) residuals instead of least squares residuals. Theil's BLUS-test statistic also has an F-distribution under H_o, which is its only simplicity (see Chaubey 1981).

Chaubey (1981) proposed a test for testing H_o using the average of the squared least squares residuals. He named the test as ASR-test. ASR-test is found to be invariant under some important transformations and the power function of the ASR-test depends on the ratio $w = \sigma_1^2 / \sigma_2^2$.

ASR-test is locally more powerful though F-test is found to be reasonable for large samples.

If we consider $k(>2)$ sources instead of two then the problem in general is to compare the variances of $k(>2)$ linear models. Harrison and McCabe (1979) suggested a test to compare k-variances and called it "Bounds test". The test statistic used by Harrison and McCabe is similar to that of ASR-test statistic though in a more general context. The problem of $k(>2)$ linear models may not be easy to deal with, however, we discuss this problem in Chapter IV.

A Bayesian analysis for testing H_o for the model (3.1.1) is presented in section 3.3. In section 3.4 we present an example using the Boot and DeWitt's (1960) data to which different methods have been applied and compare the F-test and Bayes-procedure. As it has not been possible to compute the confidence interval for ASR-test statistic, we could not compare ASR-test with Bayes-procedure. Since BLUS-test and F-test were compared in Chaubey (1981) and F-test showed better results than BLUS-test, we compared only F-test with the Bayes procedure. Section 3.2 describes various tests discussed above.

3.2 Some Common Tests of H_o

One simple method to test H_o is to consider the F-statistic, computed in the following way,

$$R_1 = \frac{\tilde{e}_1'\tilde{e}_1 \,/(n_1 - p)}{\tilde{e}_2'\tilde{e}_2 \,/(n_2 - p)} \tag{3.2.1}$$

where $\tilde{e}_i = [\,I_{n_i} - X_i(X_i'X_i)^{-1}X_i'\,]\,Y_i$,

$(i = 1,2)$ i.e. $\tilde{e}_i$'s are the least-squares residuals for the ith model. Obviously, this F-statistic ignores the equality of the regression parameters of the two linear models in (3.1.1). Under the null hypothesis, the test statistic in (3.2.1) has F-distribution with $(n_1 - p, n_2 - p)$ degrees of freedom.

BLUS-Test

Theil (1971) observed that the least squares residuals are not convenient for testing H_o, since the least square residuals e, in the combined model

$$Y = X\beta + \varepsilon \quad , \tag{3.2.2}$$

where $Y' = (Y_1' \vdots Y_2')$, $X' = (X_1' \vdots X_2')$, $\varepsilon' = (\varepsilon_1' \vdots \varepsilon_2')$,

have non-scalar convariance matrix under $H_o: \sigma_1^2 = \sigma_2^2 = \sigma^2$, viz.

$$D(e) = \sigma^2 Q \tag{3.2.3}$$

where $Q = (I - X(X'X)^{-1}X')$ is an idempotent matrix. Theil suggested to obtain that estimator of ε which is linear in Y, unbiased and has the minimum variance among the class of other estimators with the scalar covariance matrix $\sigma^2 I$. Such an estimator of the residual vector is obtained by choosing the estimator $\hat{\varepsilon} = CY$, where C is such that

$$CX = 0 \quad \text{and} \quad CC' = I \; . \tag{3.2.4}$$

C has at most $(n - r)$ rows where $r = \text{rank}(X)$ and the order of I in (3.2.4) is $(n - r)$. Therefore, $\hat{\varepsilon}$ represents at most $(n - r)$ components of the disturbance vector ε.

The model (3.2.2) can be partitioned as

$$\begin{bmatrix} Y_o \\ \cdots \\ Y_o^c \end{bmatrix} = \begin{bmatrix} X_o \\ \cdots \\ X_o^c \end{bmatrix} \beta + \begin{bmatrix} \varepsilon_o \\ \cdots \\ \varepsilon_o^c \end{bmatrix} \tag{3.2.5}$$

and equivalently,

$$\begin{bmatrix} Y_o \\ \cdots \\ Y_o^c \end{bmatrix} = \begin{bmatrix} X_o \\ \cdots \\ X_o^c \end{bmatrix} \hat{\beta} + \begin{bmatrix} e_o \\ \cdots \\ e_o^c \end{bmatrix} \tag{3.2.6}$$

where $\hat{\beta}$ is the least-squares estimator of β, e's are the least-squares residuals and X_o is non-singular. Assuming that X_o corresponds to the first r observations, it can be shown that

$$e_o = (X_o^c X_o^{-1})' e_o^c . \tag{3.2.7}$$

From (3.2.7) we conclude that the 1st r residuals and last $(n-r)$ residuals are equally informative about ε. So the $(n-r)$ residuals having scalar covariance matrix could be made to correspond to the last $(n-r)$ disturbances. Let $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ denote the BLUS-residuals for the first and second models resepectively. The dimensions of $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ depends on the selection of X_o. If X_o has r_1 rows of X_1 and r_2 rows of X_2, then $r = r_1 + r_2$. The statistic

$$R_2 = \frac{\hat{\varepsilon}_1' \hat{\varepsilon}_1 /(n_1 - r_1)}{\hat{\varepsilon}_2' \hat{\varepsilon}_2 /(n_2 - r_2)} \tag{3.2.8}$$

which has the F-distribution with $(n_1 - r_1 , n_2 - r_2)$ degrees of freedom, can be used to test H_o.

The ASR-Test

The least-squares residual vector e of (3.2.2) can be written as $e' = (e_1' \vdots e_2')$ where

$$e_1 = (I_{n_1} - X_1(X'X)^{-1}X_1')Y_1 - X_1(X'X)^{-1}X_2'Y_2 ,$$

and

$$e_2 = (I_{n_2} - X_2(X'X)^{-1}X_2')Y_2 - X_2(X'X)^{-1}X_1'Y_1 .$$

The ASR-test statistic for testing H_o is constructed on the basis of the above residual vectors as follows.

$$R_3 = \frac{e_1' e_1 / n_1}{e_2' e_2 / n_2} \tag{3.2.9}$$

The acceptance region for H_o: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 \geq \sigma_2^2$ based on R_3 is given as

$$R_3 \leq C \tag{3.2.10}$$

where C is such that

$$\Pr[R_3 \leq C \mid H_o] = 1 - \alpha . \tag{3.2.11}$$

3.3 The Bayesian Solution

A Bayesian appraoch for testing H_o can be given through the Bayesian inference on the ratio $w = \sigma_1^2 / \sigma_2^2$. When there is a posteriori evidence that this ratio is significantly close to one, we may accept H_o. Thus, we first derive the posterior distribution of w.

3.3.1 Posterior Distribution of w

The likelihood function of the model (3.1.1) is given by

$$L(\beta,\sigma_1^2,\sigma_2^2|Y)\,\alpha\,(\sigma_1^2)^{-\frac{n_1}{2}}(\sigma_2^2)^{-\frac{n_2}{2}}\exp\{-\tfrac{1}{2}\sum_{i=1}^{2}\frac{(Y_i - X_i\beta)'(Y_i - X_i\beta)}{\sigma_i^2}\}. \tag{3.3.1}$$

We consider the non-informative reference prior for $(\beta,\sigma_1^2,\sigma_2^2)$

$$p(\beta,\sigma_1^2,\sigma_2^2)\;\alpha\;\frac{1}{\sigma_1^2\sigma_2^2} . \tag{3.3.2}$$

The posterior distribution with respect to the above prior is given by

$$p(\beta,\sigma_1^2,\sigma_2^2|Y)\;\alpha\;L(\beta,\sigma_1^2,\sigma_2^2|Y)\cdot p(\beta,\sigma_1^2,\sigma_2^2)$$

$$\alpha\;(\sigma_1^2)^{-\frac{n_1}{2}-1}(\sigma_2^2)^{-\frac{n_2}{2}-1}\exp\{-\tfrac{1}{2}(\frac{\nu_1 s_1^2+S_1(\beta)}{\sigma_1^2}+\frac{\nu_2 s_2^2+S_2(\beta)}{\sigma_2^2})\}, \tag{3.3.3}$$

where $\nu_i = n_i - p$, $s_i^2 = \nu_i^{-1}(Y_i - X_i\hat{\beta}_i)'(Y_i - X_i\hat{\beta}_i)$,
$\hat{\beta}_i = (X_i'X_i)^{-1}X_i'Y_i$, $S_i(\beta) = (\beta - \hat{\beta}_i)'X_i'X_i(\beta - \hat{\beta}_i)$, $i = 1,2$.

Integrating out β from (3.3.3) gives the posterior distribution of σ_1^2, σ_2^2, namely,

$$p(\sigma_1^2, \sigma_2^2|Y) \propto (\sigma_1^2)^{-\frac{n_1}{2}-1}(\sigma_2^2)^{-\frac{n_2}{2}-1}\exp\{-\tfrac{1}{2}(\frac{\nu_1 s_1^2}{\sigma_1^2} + \frac{\nu_2 s_2^2}{\sigma_2^2})\}$$

$$\times \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(\frac{S_1(\beta)}{\sigma_1^2} + \frac{S_2(\beta)}{\sigma_2^2})\}\, d\beta \tag{3.3.4}$$

Using the following identity to simplify (3.3.4),

$$S_1(\beta) + wS_2(\beta) = S(\beta|w) + D(w) \tag{3.3.5}$$

where

$$S(\beta|w) = (\beta - \hat{\beta}_w)'(X_1'X_1 + wX_2'X_2)(\beta - \hat{\beta}_w),$$

$$D(w) = w(\hat{\beta}_1 - \hat{\beta}_2)'X_1'X_1(X_1'X_1 + wX_2'X_2)^{-1}X_2'X_2(\hat{\beta}_1 - \hat{\beta}_2)$$

and

$$\hat{\beta}_w = (X_1'X_1 + wX_2'X_2)^{-1}(X_1'X_1\hat{\beta}_1 + wX_2'X_2\hat{\beta}_2).$$

Using (3.3.5) in (3.3.4) gives,

$$p(\sigma_1^2, \sigma_2^2|Y) \propto (\sigma_1^2)^{-\frac{n_1}{2}-1}(\sigma_2^2)^{-\frac{n_2}{2}-1}\exp\{-\tfrac{1}{2}(\frac{\nu_1 s_1^2 + w\nu_2 s_2^2 + D(w)}{\sigma_1^2})\}$$

$$\times \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}\frac{S(\beta|w)}{\sigma_1^2}\}\, d\beta,$$

$$\propto (\sigma_1^2)^{-\frac{n_1}{2}}(\sigma_2^2)^{-\frac{n_2}{2}-1}|X_1'X_1 + wX_2'X_2|^{-\frac{1}{2}}$$

$$\times \exp\{-\tfrac{1}{2}\frac{\nu_1 s_1^2 + w\nu_2 s_2^2 + D(w)}{\sigma_1^2}\}. \tag{3.3.6}$$

Making the transformtion from $(\sigma_1^2, \sigma_2'^2)$ to (σ_1^2, w) as in section 2.3, and integrating out σ_1^2 from (3.3.6) gives the posterior distribution of w as

$$p(w|Y) = Kw^{\frac{n_2}{2}-1} |X_1'X_1 + wX_2'X_2|^{-\frac{1}{2}} [s^2(w)]^{-\frac{n_1+n_2-p}{2}}$$

where the constant K is determined from

$$K^{-1} = \int_{-\infty}^{\infty} w^{\frac{n_2}{2}-1} |X_1'X_1 + wX_2'X_2|^{-\frac{1}{2}} [s^2(w)]^{-\frac{n_1+n_2-p}{2}} dw \quad (3.3.7)$$

and

$$s^2(w) = (\nu_1 s_1^2 + w\nu_2 s_2^2 + D(w))/(n_1 + n_2 - p).$$

This distribution is derived in Box and Tiao (1973) to be used in computation of the posterior distribution of any subset of β and is simplified as

$$p(w|Y) = K|X_1'X_1|^{-\frac{1}{2}} [\prod_{j=1}^{p}(1+w\lambda_j)]^{-\frac{1}{2}} w^{\frac{n_2}{2}-1} [s^2(w)]^{-\frac{n_1+n_2-p}{2}},$$

for $$0 < w < \infty, \quad (3.3.8)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are diagonal elements of a matrix Λ given by $\Lambda = P'X_2'X_2P$ where P is such that $P'X_1'X_1P = I$. Computation of $\hat{\beta}_w$ and $s^2(w)$ now simplifies to

$$\hat{\beta}_w = P(I + w\Lambda)^{-1}(\hat{\phi}_1 + w\hat{\phi}_2), \quad \hat{\phi}_i = P^{-1}\hat{\beta}_i, \quad i = 1,2$$

and

$$s^2(w) = (n_1+n_2-p)^{-1}[\nu_1 s_1^2 + w\nu_2 s_2^2 + \sum_{j=1}^{p} \frac{\lambda_j}{1+w\lambda_j}(\hat{\phi}_{1j} - \hat{\phi}_{2j})^2].$$

To replace the improper integral in (3.3.7), we make the transformation $z = \frac{w}{1+w}$ and the constant K is given by

$$K^{-1} = |X_1'X_1|^{-\frac{1}{2}} \int_0^1 \Big[\big(\prod_{j=1}^{p}(1+w\lambda_j)\big)^{-\frac{1}{2}} w^{\frac{n_2}{2}} \times \{s^2(w)\}^{-\frac{n_1+n_2-p}{2}} \Big]_{w=\frac{z}{1-z}} (1-z)^{-2} dz \qquad (3.3.9)$$

where $[\cdot]_{w=\frac{z}{1-z}}$ expresses the fact that the expression in square brackets is evaluated for $w = \frac{z}{1-z}$. Computational aspects of this is addressed in section 3.4. HPD intervals based on $p(z|Y)$, $p(w|Y)$ and $p(u = \log w|Y)$ for Boot and DeWitt's data are obtained using the algorithm described in Chapter I and are presented in the next section. The posterior densities $p(z|Y)$, $p(w|Y)$ and $p(u = \log w|Y)$ are obtained through the procedures discussed in section 2.3.2. When the regression parameter β is not known to be common apriori, the HPD intervals can be obtained using F-distribution (see Box and Tiao (1973), pp. 141).

3.4 An example

We consider the time series data reported in Boot and DeWitt (1960), relating to two large companies, General Electric (GE) and Westinghouse (W)*. In this model, price deflated gross investment (I) is assumed to be a linear function of expected profitability (P) and real captial stock at the beginning of the year (S). As in Grunfeld (1958) the value of outstanding shares at the beginning of the year is taken as a measure of a company's expected profitability. The two investment relations are:

* This example is taken from Box and Tiao (1973).

$$Y_{1j} = \beta_{01} + \beta_1 X_{11j} + \beta_2 X_{12j} + \varepsilon_{1j} \ ,$$
$$Y_{2j} = \beta_{02} + \beta_1 X_{21j} + \beta_2 X_{22j} + \varepsilon_{2j} \ ,$$

where j denotes the index of a year , $j = 1, 2, \ldots, 20$ and variables are identified as,

	Company	
<u>variable</u>	<u>GE</u>	<u>W</u>
I	Y_{1j}	Y_{2j}
P	X_{11j}	X_{21j}
S	X_{12j}	X_{22j}
Error	ε_{1j}	ε_{2j} .

Performing similar analysis as in section (3.3), we can eliminate β_{01}, β_{02} and represent the posterior of $(\beta_1, \beta_2, \sigma_1^2, \sigma_2^2)$ in the form (3.3.3) (see Box and Tiao (1973), pp. 496-497) where,

$$X_1'X_1 = 10^6 \begin{bmatrix} 3.254 & 0.233 \\ 0.233 & 1.193 \end{bmatrix} , \ X_2'X_2 = 10^6 \begin{bmatrix} 0.940 & 0.195 \\ 0.195 & 0.074 \end{bmatrix} ,$$

$$\hat{\beta}_1 = \begin{bmatrix} .02655 \\ .15170 \end{bmatrix} , \qquad \hat{\beta}_2 = \begin{bmatrix} .05289 \\ .09241 \end{bmatrix} ,$$

$s_1^2 = 0.777 \times 10^3$, $s_2^2 = 0.104 \times 10^3$, $n_1 = n_2 = 19$, $p = 2$, $\nu_1 = \nu_2 = 17$. The matrices P, Λ and vectors $\hat{\phi}_1$ and $\hat{\phi}_2$ are

$$P = 10^{-3}\begin{bmatrix} -.19814 & .52196 \\ .89469 & .22306 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} .02703 & 0 \\ 0 & .30513 \end{bmatrix}$$

$$\hat{\phi}_1 = P^{-1}\hat{\beta}_1 = 10^3\begin{bmatrix} .14331 \\ .10527 \end{bmatrix}, \quad \hat{\phi}_2 = P^{-1}\hat{\beta}_2 = 10^3\begin{bmatrix} .07128 \\ .12839 \end{bmatrix}.$$

The modes of $p(w|Y)$ and $p(u = \log w|Y)$ are obtained by solving $h(w) = 0$ and $h^*(w) = 0$ iteratively (or otherwise) respectively, where

$$h(w) = (n_2 - 2)w^{-1} - \sum_j \lambda_j (1+w\lambda_j)^{-1} - [s^2(w)]^{-1}$$
$$\times [\nu_2 s_2^2 + \sum_j \lambda_j (1+w\lambda_j)^{-2} (\hat{\phi}_{1j} - \hat{\phi}_{2j})^2] \;;$$
$$h^*(w) = n_2 w^{-1} - \Sigma\lambda_j (1+w\lambda_j)^{-1} - [s^2(w)]^{-1}$$
$$\times [\nu_2 s_2^2 + \sum_j \lambda_j (1+w\lambda_j)^{-2} (\hat{\phi}_{1j} - \hat{\phi}_{2j})^2] \,,$$

FORTRAN source listing for this purpose is provided in Appendix A3.1 (subroutines SOLMOD and SOLMOD1). For this data the mode of $p(w|Y)$ and $p(u|Y)$ are found to be at $\hat{w} = 5.8586$ and $\hat{w}^* = 7.349$ respectively. The subroutine DCADRE from IMSL Library was used to compute the constant (3.3.8). The value of K is obtained to be

$$K^{-1} = .1385105765137285 \times 10^{-43} \,.$$

The posterior pdf of z is given in figure 3.1 and that of w and u are given in figures 3.2 and 3.3 respectively.

HPD - intervals were obtained using the algorithm given in Chapter I. These are given in Table 3.1. For each c the roots u_{1c}, u_{2c} were obtained using the subroutine ZFALSE from IMSL

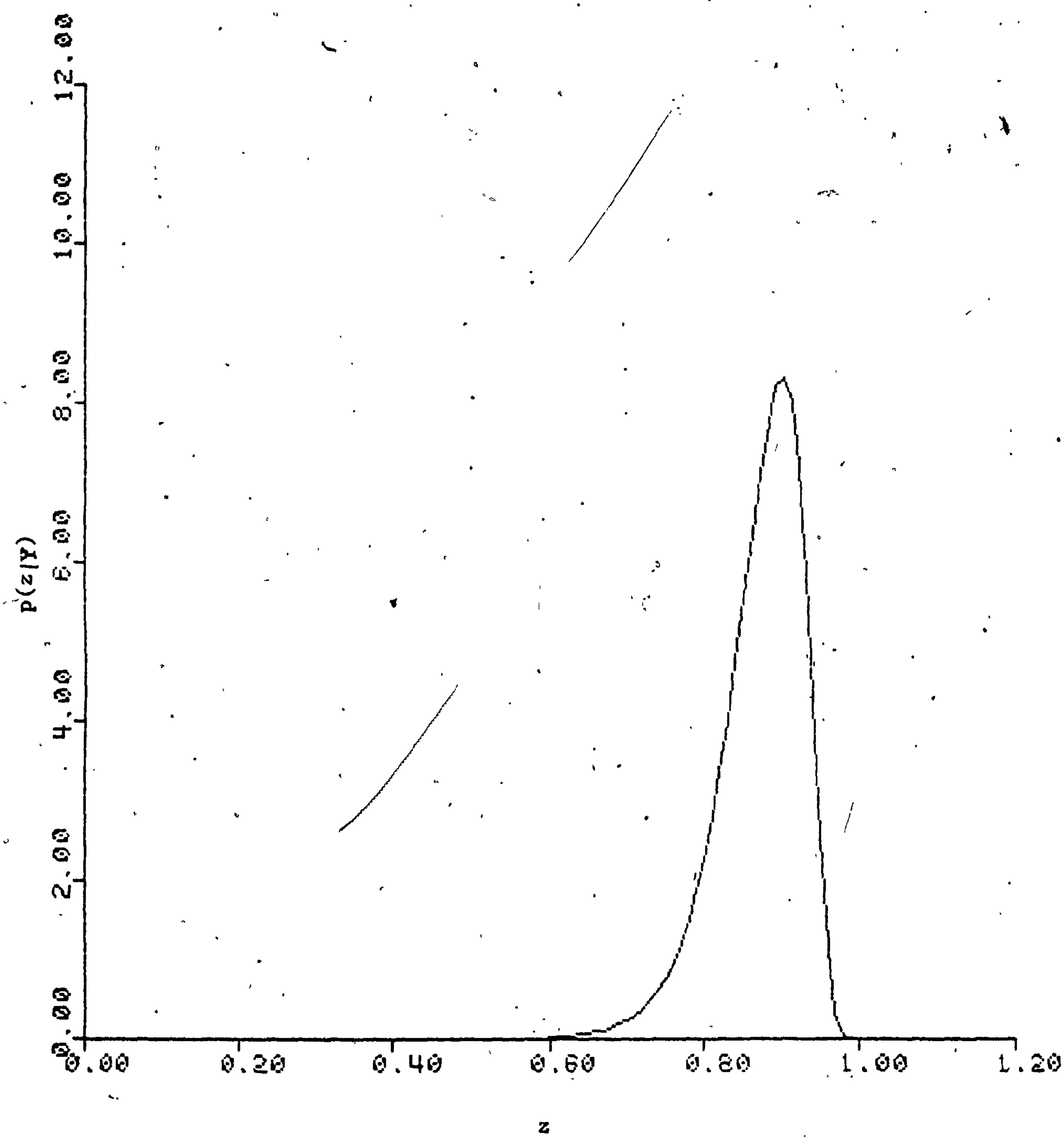


Figure 3.1 POSTERIOR DISTRIBUTION OF z GIVEN Y .

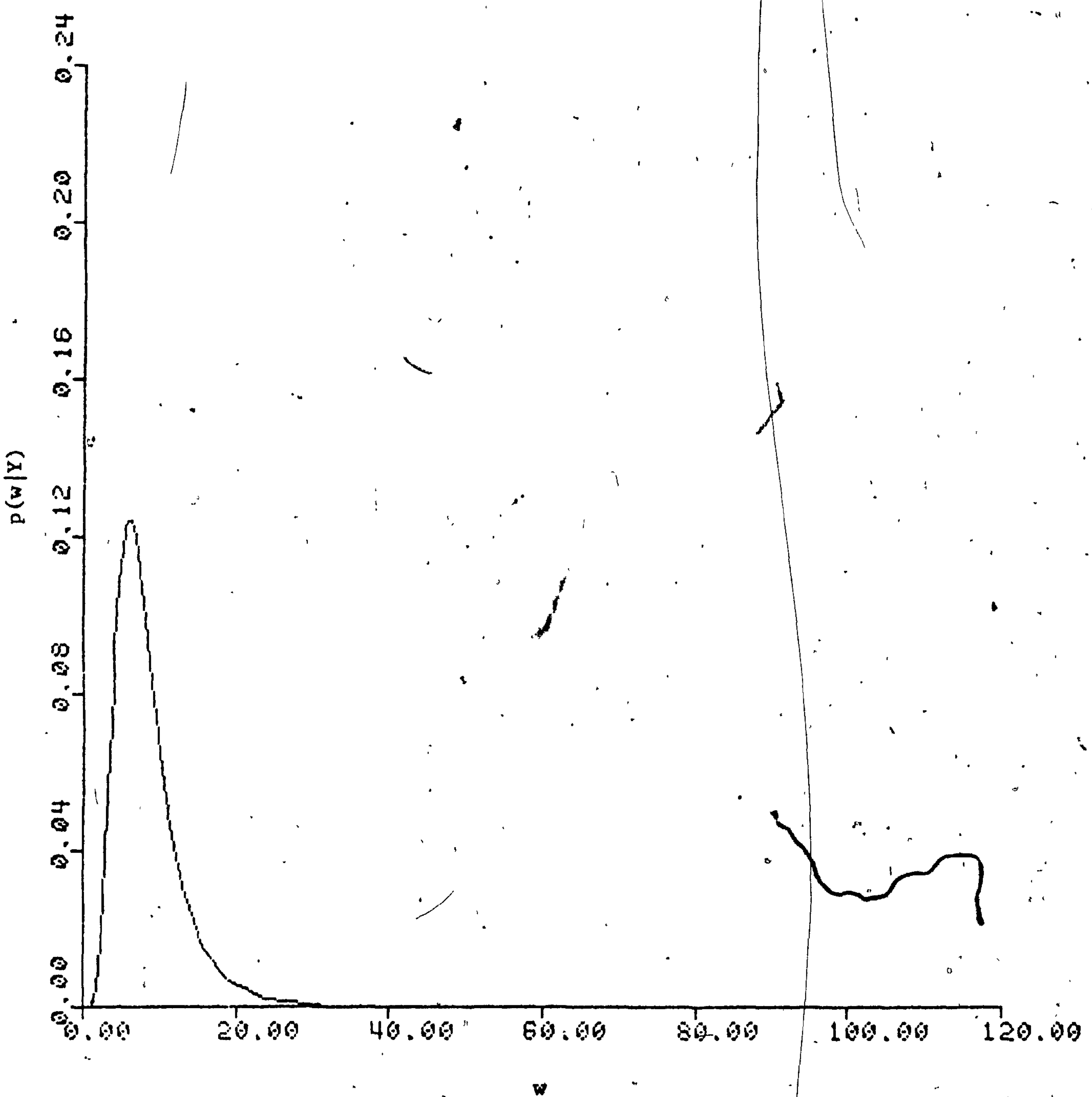


Figure 3.2 POSTERIOR DISTRIBUTION OF w GIVEN Y .

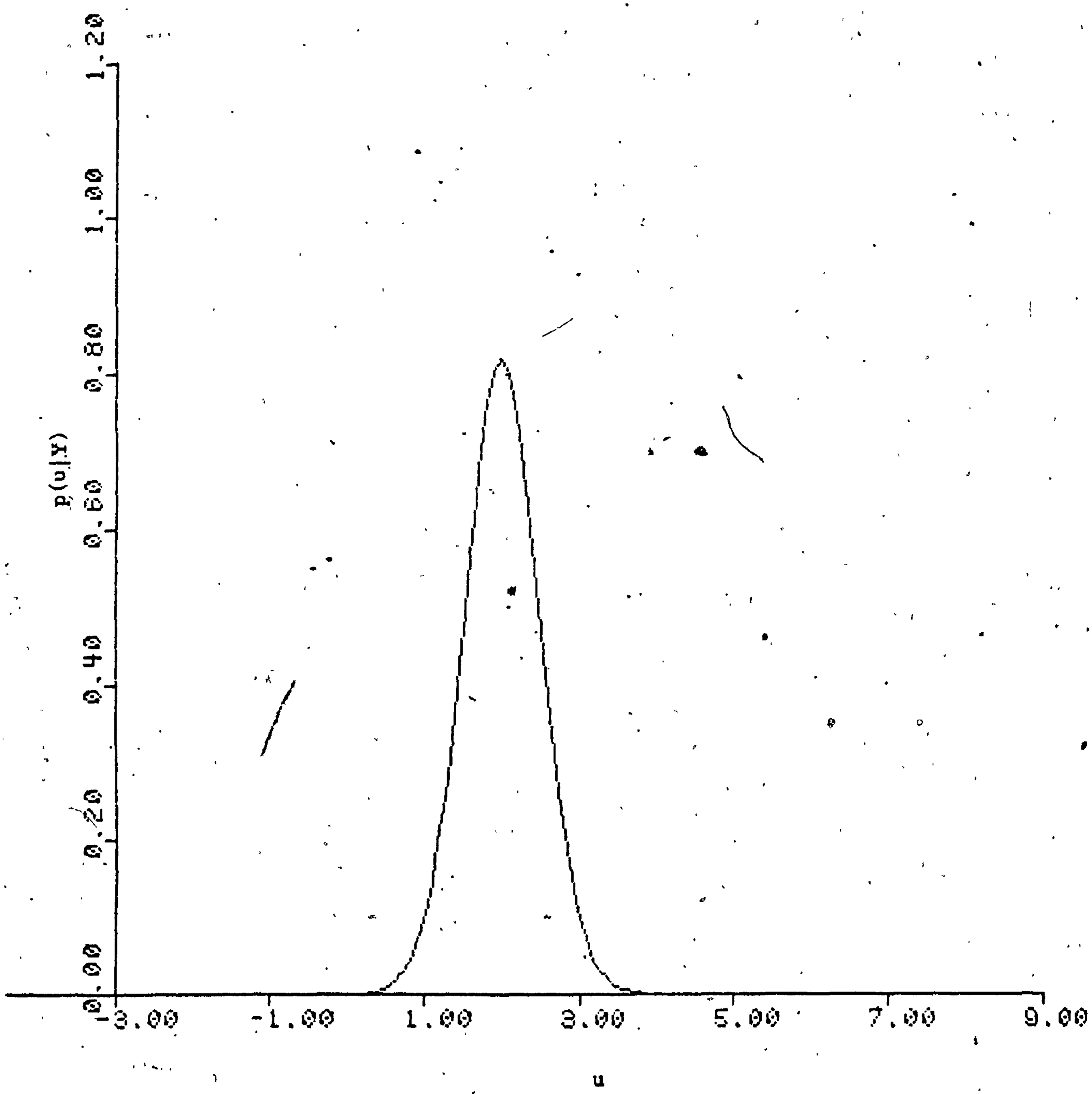


Figure 3.3 POSTERIOR DISTRIBUTION OF u GIVEN Y .

TABLE 3.1: HPD INTERVALS FOR w IN

BOOT AND DEWITT'S DATA (1960)

HPD INTERVALS

	Based on w			Based on u			Based on F		
α	Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width
.01	1.38	23.69	22.31	2.06	26.58	24.53	2.02	27.70	25.68
.05	1.98	16.83	14.83	2.83	19.28	16.46	2.79	19.97	17.18
.10	2.37	14 18	11.81	3.30	16.45	13.15	3.29	16.98	13.69
.25	3.13	10.88	7.75	5.21	12.86	8.65	4.23	13.19	8.97

TABLE 3.2: EQUAL TAIL BAYESIAN INTERVALS

FOR w IN BOOT AND DEWITT'S DATA

BAYESIAN INTERVALS (EQUAL TAILS)

	Based on u			Based on F		
α	Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width
.01	2.07	26.60	24.54	2.02	27.69	25.68
.05	2.83	19.32	16.49	2.80	19.97	17.17
.10	3.31	16.47	13.15	3.29	16.97	13.68
.25	4.22	12.88	8.66	4.23	13.12	8.99

Library. The program is given in Appendix A3.1. We also evaluate equal tail Bayesian intervals and these are given in table 3.2, which seem not to be too far off from the HPD - intervals.

Two points become evident in the analysis of the given data:

i) HPD - intervals based on w and u are shorter than those based on F.

ii) The ratio of two variances is concentrated far away from one.

CHAPTER IV

TESTING EQUALITY OF VARIANCES IN $k(\geq 2)$ LINEAR MODELS

4.1 Introduction

Let us consider $k(\geq 2)$ linear models with common β parameter

$$Y_i = X_i\beta + \varepsilon_i , \quad i = 1,2,3,\ldots,k \tag{4.1.1}$$

where Y_i is an $(n_i \times 1)$ vector of observations, X_i is a known $(n_i \times p)$ matrix with rank p and $p \leq n_i$, β is a $p \times 1$ vector of parameters, ε_i is disturbance vector of order $(n_i \times 1)$ for the ith model with

$$E(\varepsilon_i) = 0 , \quad V(\varepsilon_i) = \sigma_i^2 I_{n_i} , \quad \text{cov}(\varepsilon_i,\varepsilon_j) = 0 ; \quad i \neq j$$

and assumed to be normally distributed. The situations where such a case of linear models occurs is discussed in previous chapters (also see Box and Tiao (1973), pp. 478). We are interested in testing the hypothesis $H_o: \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$. The problem of testing H_o has been considered by many researchers (see Chaubey 1981, and Harrison and McCabe (1979) and references therein). The test proposed by Harrison and McCabe (1979) divides the observations into two groups following the suggestion of Goldfeld and Quandt (1965), though the test statistic is different from the test statistic proposed by Goldfeld and Quandt. The Bayesian analysis for this k-linear model case is difficult. However, a simple way to get a Bayesian solution for this case is to divide the observations into two groups and apply the procedures discussed in section 3.3. The other way is to generalize the likelihood ratio statistic for testing the homogeneity of variances in normal populations, which has some Bayesian support (see Box and Tiao 1973, pp. 134). In this spirit we consider the proposal of Chaubey (1980) to use a test similar to likelihood ratio test, which is based on ordinary least squares residuals. The test statistic proposed here is computationally simpler than that given by the standard likelihood ratio test. We have studied the properties of the test for two normal populations with common mean as a special case of (4.1.1) and give results for general case $(k > 2)$. It is observed that the proposed test statistic has an approximate chi-square distribution with $(k-1)$ degrees of freedom. The present test is also computationally simpler than some of the other tests proposed in literature. The next section (4.2) gives the tests proposed by Goldfeld and Quandt and Harrison and McCabe and the new test is described in section 4.3. Section 4.4 presents an approximation to the distribution of the new test statistic in terms of its moments which are evaluated for the special case with $k = 2$ and studied in detail for comparison purposes.

4.2 Goldfeld and Quandt's Parametric Test

Goldfeld and Quandt's parametric test for testing H_o can be described for the combined model,

$$Y = X\beta + \varepsilon \tag{4.2.1}$$

where $Y = [Y_1' \vdots \ldots \vdots Y_k']'$, $X = [X_1' \vdots \ldots \vdots X_k']'$, $\varepsilon = [\varepsilon_1' \vdots \ldots \vdots \varepsilon_k']'$ as follows.

First the observations are arranged in ascending order by the values of the variable X_t, where X_t is the potential deflator in econometric problem. The ordering is given in terms of the second subscript of X_t indexed so that $x_{ti} \leq x_{tj}$ if and only if $i < j$, and the remaining variables are indexed so that the indexed values correspond with those of x_{ti}. This ordering is based on the presumption that variance of the observations increase with X_t. Ignoring m central observations, two separate least-squares regressions are fitted to the first $(n-m)/2$ and last $(n-m)/2$ observations, provided that the $(n-m)/2 > p$, the number of parameters to be estimated, and that the $(n-m)/2$ observations be distributed over at least p distinct points in X space, i.e. the rank of matrix with the $(n-m)/2$ observations must be at least equal to p. Denoting S_1 and S_2, the sum of squares of the residuals from the regressions based on the relatively small and relatively large values of X_t respectively, the test statistic is formed as

$$R_1 = S_2 / S_1 , \tag{4.2.2}$$

which has F-distribution with $(\frac{n-m-2p-2}{2}, \frac{n-m-2p-2}{2})$ degrees of freedom under the null hypothesis.

Harrison and McCabe's Test

Harrison and McCabe (1979) proposed the following test statistic for testing H_o,

$$b = \frac{e'Ae}{e'e} \tag{4.2.3}$$

where A is an appropriate selector matrix of order $(n \times n)$ with $m(0 < m < n)$ ones and $(n-m)$ zeros on its principal diagonal and zeros elsewhere, and e is the least squares residuals obtained from the combined model (4.2.1).

This test uses the F-distribution for finding the critical values of b, and one sided H_o would be rejected if $b < b_L^*$ and accepted if $b > b_U^*$ and the test is inconclusive if $b_L^* < b < b_U^*$; where b_L^*

and b_U^* are critical values of test statistic for a required level of significance and obtained using the α critical value of F as

$$b_L^* = [1 + \frac{(n-m)F_\alpha(n-m,m-k)}{m-k}]^{-1}$$

$$b_U^* = [1 + \frac{(n-m-k)F_\alpha(n-m-k,m)}{m}]^{-1} ,$$

where $F_\alpha(\nu_1,\nu_2)$ denotes the upper α-percentile of F-distribution with degrees of freedom (ν_1,ν_2). When the test is inconclusive, use of an exact test using an exact or an approximate distribution of b is proposed.

4.3 A Likelihood Ratio Type Test for H_0

Chaubey (1980) proposed the following likelihood ratio type test for testing H_0. This test is motivated by the likelihood ratio test criterion (Λ) for testing equality of variances of k normal populations, which has the form

$$\Lambda = \prod_{i=1}^{k} (\hat{\sigma}_i^2)^{n_i/2} / (S^2)^{n/2} \quad , \tag{4.3.1}$$

where $\hat{\sigma}_i^2$ is the maximum likelihood estimate of σ_i^2 and S^2 is the maximum likelihood estimate of σ^2 (the common variance under the null hypothesis). Since the computation of $\hat{\sigma}_i^2$'s are difficult in the present case it is replaced by

$$\tilde{\sigma}_i^2 = e_i' e_i / n_i$$

where $e_i = Y_i - X_i\hat{\beta}$ and $\hat{\beta}$ is the ordinary least-squares estimate of β for the combined model given in (4.2.1) and substituting the maximum likelihood estimator $(\hat{\sigma}^2 = e'e/n)$ in place of S^2 in (4.3.1) we obtain

$$T = -2\ln \Lambda = -\sum_{i=1}^{k} n_i \ln(e_i' e_i / e'e) + \sum_{i=1}^{k} n_i \ln(n_i/n) \quad . \tag{4.3.2}$$

The test statistic T for large samples has approximately a chi-square distribution with $(k-1)$ degrees of freedom, even though $\tilde{\sigma}_i^2$ are not individual MLE's of σ_i^2. The exact distribution of T is very difficult to obtain and we shall instead obtain an approximation to its distribution as follows.

4.4 Approximation to the Distribution of T Under H_o

We approximate T by a constant multiple of chi-square variable, because T is a linear function involving the variables $(e_i'e_i/e'e)$ which behaves like a beta variable and negative logarithm of beta variable behaves like a chi-square variable (see Wise (1950)). Thus letting $T \sim a\chi_\nu^2$ and equating the first two moments we get,

$$a = \mu_2/2\mu_1' \ , \quad \nu = 2\mu_1'/\mu_2 \ , \tag{4.4.1}$$

where μ_1' and μ_2 represent the mean and variance of T respectively. These are not explicitly obtainable. However the approximate moments can be obtained as in the following section.

4.4.1 Approximate Moments of T

Note from (4.3.2) that

$$T = g(\underset{\sim}{b}) + \sum_{i=1}^{k} n_i \ln(n_i/n) \ ,$$

where $\underset{\sim}{b} = (b_1, b_2, \ldots, b_{k-1})$, $b_i = e_i'e_i/e'e$

and
$$g(\underset{\sim}{b}) = -\sum_{i=1}^{k-1} n_i \ln(b_i) - n_k \ln(1 - \sum_{i=1}^{k-1} b_i) . \tag{4.4.2}$$

We find the approximate moments of T by expanding $g(\underset{\sim}{b})$ in Taylor's series about $\underset{\sim}{\theta} = E(\underset{\sim}{b})$,

$$g(\underset{\sim}{b}) \simeq g(\underset{\sim}{\theta}) + \sum_{i=1}^{k-1} (b_i - \theta_i) \left.\frac{\partial g(\underset{\sim}{b})}{\partial b_i}\right|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \frac{1}{2!} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (b_i - \theta_i)(b_j - \theta_j) \left.\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j}\right|_{\underset{\sim}{b}=\underset{\sim}{\theta}} \tag{4.4.3}$$

Taking expectation on both sides of (4.4.3) we get,

$$\delta_1 = E(g(\underset{\sim}{b})) \approx g(\underset{\sim}{\theta}) + \frac{1}{2} \sum_{i}^{k-1} \sum_{j}^{k-1} \mu_{ij} \left. \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j} \right|_{\underset{\sim}{b}=\underset{\sim}{\theta}} \qquad (4.4.4)$$

where

$$\mu_{ij} = E(b_i - \theta_i)(b_j - \theta_j) .$$

Also from (4.4.3),

$$\delta_2 = E(g(\underset{\sim}{b}) - g(\underset{\sim}{\theta}))^2 \approx \sum_{i \neq j}\sum E(b_i - \theta_i)(b_j - \theta_j)\left(\frac{\partial g(\underset{\sim}{b})}{\partial b_i} \cdot \frac{\partial g(\underset{\sim}{b})}{\partial b_j}\right|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \sum_{i \neq j \neq l}\sum\sum E(b_i - \theta_i)(b_j - \theta_j)(b_l - \theta_l)\left(\frac{\partial g(\underset{\sim}{b})}{\partial b_i} \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_j \partial b_l}\right)\Bigg|_{b=\theta}$$

$$+ \sum_{i} E(b_i - \theta_i)^2 \left(\frac{\partial g(b)}{\partial b_i}\right)^2 \Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \sum_{i \neq j}\sum E(b_i - \theta_i)^2 (b_j - \theta_j)\left(\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i^2} \cdot \frac{\partial g(\underset{\sim}{b})}{\partial b_j} + \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j} \cdot \frac{\partial g(\underset{\sim}{b})}{\partial b_i}\right)\Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \frac{1}{4} \sum_{i \neq j \neq l}\sum\sum E(b_i - \theta_i)^2 (b_j - \theta_j)(b_l - \theta_l)\left(\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j} \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j}\right.$$

$$\left. + 2 \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i^2} \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_j \partial b_l}\right)\Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \frac{1}{4} \sum_{i \neq j}\sum E(b_i - \theta_i)^2 (b_j - \theta_j)^2 \left(\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i^2} \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_j^2} + \left(\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j}\right)^2\right)\Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \sum_{i} E(b_i - \theta_i)^3 \left(\frac{\partial g(\underset{\sim}{b})}{\partial b_i} \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i^2}\right)\Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \frac{1}{2} \sum_{i \neq j}\sum E(b_i - \theta_i)^3 (b_j - \theta_j) \left(\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i^2} \cdot \frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i \partial b_j} \right) \Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}}$$

$$+ \frac{1}{4} \sum_i E(b_i - \theta_i)^4 \left(\frac{\partial^2 g(\underset{\sim}{b})}{\partial b_i^2} \right)^2 \Bigg|_{\underset{\sim}{b}=\underset{\sim}{\theta}} . \qquad (4.4.5)$$

Ordinarily, approximations for δ_1 and δ_2 may be appropriate by considering only the terms with first derivative and ignoring the terms with second and higher order derivatives, but at least in one situation it is not so, as we will see in section 4.4.3. That is why we retained terms with second derivatives in (4.4.3) and (4.4.5). Using these approximations we can obtain μ_1' and μ_2 to be used in (4.4.1) as

$$\mu_1' = \delta_1 + \Sigma\, n_i \ln(n_i/n) , \qquad (4.4.6)$$

$$\mu_2 = \delta_2 - \delta_1^2 . \qquad (4.4.7)$$

Computation of δ_1 and δ_2 in (4.4.4) and (4.4.5) involve the moments, cross-moments and cross-moments of higher powers of b_i, b_j, b_l which are obtained in the following section.

4.4.2 Computation of Moments, Cross-Moments and Cross-Moments of Higher Powers of b_i, b_j, b_l

Since b_i's are independent of $e'e$ by Geary's Theorem (1933), the moments of b_i's are easily obtained as

$$\mu_r' = E(b_i^r) = \frac{E(e_i'e_i)^r}{E(e'e)^r} . \qquad (4.4.8)$$

The raw moments in the numerator and the denominator of (4.4.8) are obtained from the following formula for the rth cumulant k_r of a quadratic form in normal variables with zero mean and covariance matrix Σ (see Searle, 1971, pp. 56, Corollary 1.1),

$$k_r = (r-1)!\, 2^{r-1}\, \mathrm{tr}(A\Sigma)^r \qquad (4.4.9)$$

and using the following relation (see Chaubey and Talukder, 1983, Lemma 1, pp. 676-677) successively between the raw moments m'_r and the cumulants k_r,

$$m'_{r+1} = \sum_{i=0}^{r} \binom{r}{i} m'_{r-i}\, k_{i+1}, \quad r = 0, 1, 2, \ldots \qquad (4.4.10)$$

The cross moments between b_i, b_j, b_l and other cross moments of higher order can be obtained using Geary's result as

$$E(b_i^r b_j^s b_l^t) = \frac{E(e_i'e_i)^r (e_j'e_j)^s (e_l'e_l)^t}{E(e'e)^{r+s}} \qquad (4.4.11)$$

where r, s, t take values from the set $(0, 1, 2, 3)$. Noting that $e = Q\varepsilon$, $e_i'e_i = e'A_ie$, where $Q = I - X(X'X)^{-1}X'$, and A_i is Block-diagonal matrix with ith Block being I_{n_i} and zero elsewhere, the denominator of (4.4.11) can be evaluated using (4.4.9) and the expectations in the numerator can be evaluated using the results of Magnus (1978) quoted below:

<u>Theorem:</u>

Let A, B and C be symmetric matrices of order n and $\varepsilon \sim N(0,\Sigma)$ where Σ is positive definite, then

i) $E(\varepsilon'A\varepsilon \cdot \varepsilon'B\varepsilon) = (\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, B\Sigma) + 2\, \mathrm{tr}\, A\Sigma B\Sigma)$.

ii) $$E(\varepsilon'A\varepsilon \cdot \varepsilon'B\varepsilon \cdot \varepsilon'C\varepsilon) = (\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, C\Sigma) + 2[(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, B\Sigma C\Sigma) + (\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, A\Sigma C\Sigma) + (\mathrm{tr}\, C\Sigma)(\mathrm{tr}\, A\Sigma B\Sigma)] + 8\, \mathrm{tr}\, A\Sigma B\Sigma C\Sigma .$$

iii) $$E[(\varepsilon'A\varepsilon)^2 \cdot (\varepsilon'B\varepsilon)] = (\mathrm{tr}\, A\Sigma)^2(\mathrm{tr}\, B\Sigma) + 4(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, A\Sigma B\Sigma) + 2(\mathrm{tr}\, B\Sigma)\mathrm{tr}(A\Sigma)^2 + 8\, \mathrm{tr}(A\Sigma)^2 B\Sigma .$$

iv) $E[(\varepsilon' A\varepsilon)^2 \cdot (\varepsilon' B\varepsilon)^2] = (\mathrm{tr}\, A\Sigma)^2(\mathrm{tr}\, B\Sigma)^2 + 16[(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, A\Sigma(B\Sigma)^2)$

$+ (\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, (A\Sigma)^2 B\Sigma)]$

$+ 4[(\mathrm{tr}(A\Sigma)^2)(\mathrm{tr}(B\Sigma)^2) + 2(\mathrm{tr}\, A\Sigma B\Sigma)^2]$

$+ 2[(\mathrm{tr}\, A\Sigma)^2 \cdot (\mathrm{tr}(B\Sigma)^2) + 4(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, A\Sigma B\Sigma)$

$+ (\mathrm{tr}\, B\Sigma)^2(\mathrm{tr}\, (A\Sigma)^2)]$

$+ 16[\mathrm{tr}(A\Sigma B\Sigma)^2 + 2\mathrm{tr}\, (A\Sigma)^2(B\Sigma)^2]$.

v) $E[(\varepsilon' A\varepsilon)^2 \cdot (\varepsilon' B\varepsilon) \cdot (\varepsilon' C\varepsilon)] = (\mathrm{tr}\, A\Sigma)^2(\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, C\Sigma)$

$+ 8\ [(\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, (A\Sigma)^2 C\Sigma) + (\mathrm{tr}\, C\Sigma)(\mathrm{tr}(A\Sigma)^2 B\Sigma) + 2(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, A\Sigma B\Sigma C\Sigma)]$

$+ 4\ [(\mathrm{tr}\, (A\Sigma)^2 \mathrm{tr}\, (B\Sigma C\Sigma) + (\mathrm{tr}\, A\Sigma B\Sigma)(\mathrm{tr}\, A\Sigma C\Sigma) + (\mathrm{tr}\, A\Sigma C\Sigma)(\mathrm{tr}\, A\Sigma B\Sigma)]$

$+ 2\ [(\mathrm{tr}\, A\Sigma)^2(\mathrm{tr}\, B\Sigma C\Sigma) + 2(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, A\Sigma C\Sigma)$

$+ 2(\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, C\Sigma)(\mathrm{tr}\, A\Sigma B\Sigma) + (\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, C\Sigma)(\mathrm{tr}\, (A\Sigma)^2)]$

$+ 16[\mathrm{tr}\, (A\Sigma)^2 B\Sigma C\Sigma + \mathrm{tr}\, (A\Sigma)^2\, C\Sigma B\Sigma + \mathrm{tr}\, (A\Sigma B\Sigma A\Sigma C\Sigma)]$

vi) $E[(\varepsilon' A\varepsilon)^3 \cdot \varepsilon' B\varepsilon] = (\mathrm{tr}\, A\Sigma)^3(\mathrm{tr}\, B\Sigma)$

$+ 24\ (\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, (A\Sigma)^2 B\Sigma) + 8\ (\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, (A\Sigma)^3$

$+ 12(\mathrm{tr}\, (A\Sigma)^2)(\mathrm{tr}\, A\Sigma B\Sigma) + 6[(\mathrm{tr}\, A\Sigma)^2(\mathrm{tr}\, A\Sigma B\Sigma)$

$+ (\mathrm{tr}\, A\Sigma)(\mathrm{tr}\, B\Sigma)(\mathrm{tr}\, (A\Sigma)^2)]$

$+ 16\ [\ 2\mathrm{tr}\, (A\Sigma)^3 B\Sigma + \mathrm{tr}\, (A\Sigma)^2 B\Sigma A\Sigma\ ]$.

4.4.3 A Special Case

In this section we consider the special case of (4.1.1) where $k = 2$, $X_1 = \underset{\sim}{1}_{n_1}$, $X_2 = \underset{\sim}{1}_{n_2}$, $Q = (I - \frac{1}{n} E_{n\times n})$, $A_1 = \begin{bmatrix} I_{n_1} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_2} \end{bmatrix}$ and $E_{n\times n}$ is $n\times n$ matrix all of whose elements are unity, I_{n_i} is the identity matrix of order n_i and $\underset{\sim}{1}_{n_i}$, is a $(n_i \times 1)$, $(i = 1,2)$ vector of elements unity. Partitioning Q as

$$Q = \left[\begin{array}{c:c} I_{n_1} - \frac{1}{n} E_{n_1 \times n_1} & -\frac{1}{n} E_{n_1 \times n_1} \\ \hdashline -\frac{1}{n} E_{n_2 \times n_1} & I_{n_2} - \frac{1}{n} E_{n_1 \times n_2} \end{array}\right]$$

and multiplying by A_1 and A_2 respectively, we get.

$$A_1 Q = \left[\begin{array}{c:c} I_{n_1} - \frac{1}{n} E_{n_1 \times n_1} & -\frac{1}{n} E_{n_1 \times n_2} \\ \hdashline 0 & 0 \end{array}\right] \tag{4.4.12}$$

$$A_2 Q = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline -\frac{1}{n} E_{n_2 \times n_1} & I_{n_2} - \frac{1}{n} E_{n_2 \times n_2} \end{array}\right] \tag{4.4.13}$$

In this case $g(b_1) = -n_1 \ln b_1 - n_2 \ln(1 - b_1)$. The Taylor's series expansion of $g(b_1)$ about $\theta_1 = E(b_1) = n_1/n$ gives as a special case of (4.4.3), since

$$g'(\theta_1) = -\frac{n_1}{\theta_1} + \frac{n_2}{1-\theta_1} = 0$$

$$\delta_1 \simeq g(\theta_1) + \frac{1}{2} m_2 g''(\theta_1) \ , \tag{4.4.14}$$

$$\delta_2 \simeq \frac{1}{4} m_4 [g''(\theta_1)]^2 \ . \tag{4.4.15}$$

Applying the method discussed in section 4.2 we find the central moments $m_i (i = 1,2,3,4)$ of b_1 which are,

$$m_1 = \frac{n_1}{n} \ ,$$

$$m_2 = \frac{2n_1(n-2)(n-n_1)}{n^2(n^2-1)} \ ,$$

$$m_3 = \frac{8n_1(n-3)(n-2n_1)(n-n_1)}{n^3(n^2-1)(n+3)} ,$$

and $$m_4 = \frac{12n_1(n-n_1)[n_1(n-n_1)(n-6)(n-10)+4n^2(n-1)]}{n^4(n^2-1)(n+3)(n+5)} .$$

Using (4.4.6) and (4.4.7) we get

$$\mu_1' = \frac{n(n-2)}{(n^2-1)} , \qquad (4.4.16)$$

and

$$\mu_2 = \frac{12n^4(n-4)}{n_1 n_2(n^2-1)(n+3)(n+5)} + \frac{3n^2(n-6)(n-10)}{(n^2-1)(n+3)(n+5)} . \qquad (4.4.17)$$

It is observed (see table 4.1) that as the sample size n increases the mean μ_1' and the variance μ_2 of T approaches to 1 and 2 respectively, and also the constants of approximation a and ν in (4.4.1) approach to unity which shows the justification of the chi-square approximation to T.

Table - 4.1: Constants of Approximation (4.4.1) for different n_1 and n_2

n	n_1	n_2	μ_1'	μ_2	ν	'a'
10	5	5	.808	.839	1.557	.519
11	5	6	.825	.912	1.493	.553
13	5	8	.851	1.089	1.330	.640
15	5	10	.871	1.276	1.188	.733
20	5	15	.902	1.704	.956	.944

Since μ_1' and μ_2 in (4.4.16) and (4.4.17) are symmetric in n_1 and n_2 we get the same value for μ_1', μ_2, ν and a as in table 4.1 if we increase n_1 keeping n_2 fixed. In the next section we investigate the accuracy of this approximation.

4.4.4 Adequacy of the Approximation

In order to check the accuracy of the chi-square approximation to the distribution of T under H_o a computer simulation is performed. Approximate $100(1-\alpha)\%$ points of T denoted by $T^*_{1-\alpha} = a\chi^2_{\nu;\alpha}$, where $\chi^2_{\nu;\alpha}$ denotes the upper α percentile of chi-square distribution with ν degrees of freedom are obtained for different n_1 and n_2 by evaluating $\chi^2_{\nu;\alpha}$ through the IMSL subroutine MDCHI. We generate 5000 random values of T using the fact that

$$e_1'e_1 = U_1 + \frac{n_2}{(n_1+n_2)} U_3$$

$$e_2'e_2 = U_2 + \frac{n_1}{(n_1+n_2)} U_3$$

where U_1, U_2, U_3 are independent chi-square random variables with $n_1 - 1$, $n_2 - 1$, and 1 degrees of freedom respectively when $\sigma_1^2 = \sigma_2^2 = 1$ (see Chaubey (1981) for the proof). The relative frequency of the event $T \geq T^*_{1-\alpha}$ denoted by f_α is observed and presented in table 4.2. If the chi-square approximation to the distribution of T is reasonable one should expect $f_\alpha \approx \alpha$. It is evident from table 4.2 that the approximation is in close agreement with the true values of $T_{1-\alpha}$ even for a small value of $n = 15$. It is also observed that for $n < 15$, the approximation is not very good. Numerical calculations also indicate that as n grows large, f_α approaches closer to α.

Table 4.2: Approximate $(1-\alpha)100\%$ Points of T and a Comparison For Their Accuracy

n_1	n_2	$T^*_{.90}$	$T^*_{.95}$	$T^*_{.99}$	$f_{.10}$	$f_{.05}$	$f_{.01}$
5	5	1.9772	2.6484	4.2289	.18056	.11380	.05099
	6	2.0395	2.7430	4.4120	.16797	.10200	.04599
	8	2.16317	2.9512	4.8312	.15917	.09440	.03279
	10	2.2684	3.1440	5.2615	.14797	.08820	.02680
6	5	2.0395	2.7430	4.4120	.17397	.10280	.04099
	6	2.0796	2.8009	4.5141	.15637	.10380	.04279
	8	2.1728	2.9525	4.8106	.14577	.09420	.03139
	10	2.2606	3.1065	5.1533	.14677	.08760	.02720
8	5	2.16317	2.9512	4.8312	.15197	.08800	.03059
	6	2.1728	2.9525	4.8106	.14597	.09700	.03539
	8	2.2181	3.0162	4.9193	.14817	.09340	.02819
	10	2.2741	3.1124	5.1263	.14017	.08500	.02720
10	5	2.2684	3.1440	5.2615	.14177	.09060	.02700
	6	2.2606	3.1065	5.1533	.14257	.08440	.02480
	8	2.2741	3.1124	5.1263	.14037	.08460	.02819
	10	2.3057	3.1596	5.2130	.13097	.08180	.02540

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APPENDICES

APPENDIX A2.1

```
      PROGRAM MAIN(INPUT,OUTPUT,GRAPHD,TAPE6=GRAPHD)
C     THIS PROGRAM COMPUTES P(Z/Y),P(U/Y) AND P(W/Y)
C     FOR DIFFERENT SAMPLE SIZES M,N,SAMPLE
C     VARIANCES V1,V2 AND DIFFERENCE BETWEEN THE
C     SAMPLE MEANS D
      DIMENSION X(99),Y(99),Z(99),U(99),W(99),FU(99),FW(99)
      COMMON/BL/M,N,V1,V2,D
      PRINT 5
5     FORMAT("ENTER VALUES FOR M,N,V1,V2,D")
      READ*,M,N,V1,V2,D
      PRINT 15, M,N,V1,V2,D
15    FORMAT(" M = ",I2," N = ",I2," V1 = ",F8.4, " V2 = ",F8.4
     1, " D = ",F8.4)
      CALL CMEAN(X,Y,Z,U,W,FU,FW)
      DO 10 I=1,99
      WRITE(6,100)  X(I),Y(I),Z(I,U(I),W(I),FU(I),FW(I)
100   FORMAT(7(2X,F13.6))
10    CONTINUE
      STOP
      END
      SUBROUTINE CMEAN (X,Y,Z,U,W,FU,FW)
C     COMPUTES THE POSTERIOR DISTRIBUTION OF
C     Z,U AND W
      DIMENSION X(99),Y(99),Z(99),U(99),W(99),FU(99),FW(99)
      COMMON /BL/M,N,V1,V2,D
      EXTERNAL F
      AL=0. OEO
      AU=1. OEO
      AUX=0.01
      CONS=DCADRE(F,AL,AU,1. OE-25,1. OE-1,ERR,IER)
      I=1
10    FX=DCADRE(F,AL,AUX,1. OE-25,1. OE-1,ERR,IER)
      X(I)=AUX
      Y(I)=FX/CONS
      Z(I)=F(AUX)/CONS
      I=I+1
      AUX=AUX+0.01EO
      IF(AUX.LE..99) GO TO 10
      DO 11 J=1,99
      W(J)=X(J)/(1.-X(J))
      U(J)=ALOG(W(J))
      FW(J)=Z(J)*(1.-X(J))**2.0
      FU(J)=FW(J)*W(J)
11    CONTINUE
      RETURN
      END
      FUNCTION F(X)
      COMMON /BL/M,N,V1,V2,D
      IF(X.EQ. 0.OEO .OR. X.EQ. 1.OEO) GO TO 1
      W=X/(1-X)
      N1=M-1
      N2=N-1
      T1=(N1*V1+N2*W*V2)
      T2=M*N*W*D*D/(M+N*W)
      FP=(M+N*W)
      FP=W**((N/2.0)-1.0)*FP**(-0.5)*((1.0-X)**(-2.0))
      F=FP/(T1+T2)**((M+N-1.0)/2.0)
      GO TO 2
1     F=0.OEO
2     RETURN
      END
```

```
      PROGRAM PLOTM(INPUT,OUTPUT,TAPE99,TAPE5=INPUT,
     1TAPE6=OUTPUT)
C      THIS PROGRAM PLOTS P(Z/Y),P(U/Y)AND P(W/Y)
C      USING THE DATA GENERATED BY THE PRECEEDING
C      PROGRAM
      DIMENSION X(101),Y(101),Z(101),U(101),W(101),
     1FU(101),FW(101)
       DO 10 I=1,99
      READ(5,100)X(I),Y(I),Z(I),U(I),W(I),FU(I),FW(I)
100   FORMAT(7(2X,F13.6))
10    CONTINUE
      CALL SCALE(X,6.,99,1)
      CALL AXIS(1.5,1.5,' ',-1,6.,0.,X(100),X(101))
      CALL SCALE(Z,6.,99,1)
      CALL AXIS(1.5,1.5,' ',1,6.,90.,Z(100),Z(101))
      CALL NEWPEN(1)
      CALL PLOT(1.5,1.5,-3)
      CALL LINE(X,Z,99,1,0,0)
      CALL SYMBOL(-.29,2.7,.14,102,90.,0)
      CALL SYMBOL(-.29,2.9,.14,40,90.,0)
      CALL SYMBOL(-.29,3.1,.14,120,90.,0)
      CALL SYMBOL(-.29,3.3,.14,124,90.,0)
      CALL SYMBOL(-.29,3.5,.14,121,90.,0)
      CALL SYMBOL(-.29,3.7,.14,41,90.,0)
      CALL SYMBOL(2.95,-.6,.14,120,0.,0)
      CALL PLOT(0.,0.,999)
      CALL SCALE(U,9.,99,1)
      CALL AXIS(1.5,1.5,' ',-1,6.,0.,U(100),U(101))
      CALL SCALE(FU,6.,99,1)
      CALL AXIS(1.5,1.5,' ',1,6.,90.,FU(100),FU(101))
      CALL NEWPEN(1)
      CALL PLOT(1.5,1.5,-3)
      CALL LINE(U,FU,99,1,0,0)
      CALL SYMBOL(-.29,2.7,.14,102,90.,0)
      CALL SYMBOL(-.29,2.9,.14,40,90.,0)
      CALL SYMBOL(-.29,3.1,.14,117,90.,0)
      CALL SYMBOL(-.29,3.3,.14,124,90.,0)
      CALL SYMBOL(-.29,3.5,.14,121,90.,0)
      CALL SYMBOL(-.29,3.7,.14,41,90.,0)
      CALL SYMBOL(2.95,-.6,.14,117,0.,0.)
      CALL PLOT(0.,0.,999)
      CALL SCALE(W,48.,99,1)
      CALL AXIS(1.5,1.5,' ',-1,6.,0.,W(100),W(101))
      CALL SCALE(FW,3.,99,1)
      CALL AXIS(1.5,1.5,' ',1,6.,90.,FW(100),FW(101))
      CALL NEWPEN(1)
      CALL PLOT(1.5,1.5,-3)
      CALL LINE(W,FW,99,1,0,0)
      CALL SYMBOL(-.29,2.7,.14,102,90.,0)
      CALL SYMBOL(-.29,2.9,.14,40,90.,0)
      CALL SYMBOL(-.29,3.1,.14,119,90.,0)
      CALL SYMBOL(-.29,3.3,.14,124,90.,0)
      CALL SYMBOL(-.29,3.5,.14,121,90.,0)
      CALL SYMBOL(-.29,3.7,.14,41,90.,0)
      CALL SYMBOL(2.95,-.6,.14,119,0.,0)
      CALL PLOT(0.,0.,999)
      CALL PLOT(0.,0.,9999)
      STOP
      END
```

APPENDIX A2.2

```
      PROGRAM QDATA2(INPUT,OUTPUT,RES,TAPE1=RES)
C     THIS PROGRAM GENERATES THE SAMPLE VARIANCES
C     AND THE DIFFERENCE BETWEEN THE SAMPLE MEANS
C     FOR TWO INDEPENDENT NORMAL POPULATIONS FOR
C     GIVEN SAMPLE SIZES  AND THE RATIO W USING
C     THE FACT THAT THE SAMPLE VARIANCES ARE DISTRI-
C     BUTED AS CHISQUARES  AND DIFFERENCE BETWEEN
C     THE MEANS  IS NORMAL AND COMPUTES THE
C     HPD INTERVALS BASED ON W AND LOG(W) FOR THE
C     CORRESPONDING SAMPLE SIZES AND SAMPLE VARIANCES
C     IT ALSO OBSERVES THE NUMBER OF TIMES THE HPD-
C      INTERVALS INCLUDE THE SPECIFIED W AND THE
C     AVERAGE WIDTHS OF THE INTERVALS
       COMMON /BLI/M,N,V1,V2,D/UBL/W1C,W2C/BL1/ALPHA
     1/UBL1/W1CU,W2CU
      DOUBLE PRECISION DSEED
      DIMENSION N1(6),W(5),Z(10)
      DATA ALPHA/.05/,CMIN/.0001/,CMAX/.9999/
      DATA FMIN/.220/,FMAX/6.086/
      IFW=0
      IFU=0
      IFF=0
      WTHW=0.0
      WTHU=0.0
      WTHF=0.0
      DATA N1/5,10,15,20,30,50/,W/.1,.5,1.0,1.5,2.0/
      DO 10 I=2,2
      M=N1(I)
      DO11 J=1,1
      N=N1(J)
      DO 12 K=3,3
      L=0
      DSEED=345673651.D0
      W1=W(K)
      NU1=M-1
      NU2=N-1
      WRITE(1,101)W1,ALPHA
101   FORMAT(5X,*W1=*,F10.4,*ALPHA=*,F10.4)
15    R=GGNQF(DSEED)
      D=R*SQRT(1.0/M+1.0/(W1*N))
      CALL GGCHS(DSEED,NU1,Z,U1)
      CALL GGCHS(DSEED,NU2,Z,U2)
      V1=U1/NU1
      V2=U2/(NU2*W1)
      CALL HPDW(M ,N ,V1,V2,D,ALPHA,CMIN,CMAX,W1C,W2C)
      W1F=FMIN*V1/V2
      W2F=FMAX*V1/V2
      CALL HPDW1(M,N,V1,V2,D,ALPHA,CMIN,CMAX,W1CU,W2CU)
      IF((W1.LT.W2C) .AND. (W1.GT.W1C))IFW=IFW+1
      IF((W1.LT.W2CU).AND.(W1.GT.W1CU))IFU=IFU+1
      IF((W1.LT.W2F).AND.(W1.GT.W1F))IFF=IFF+1
      WTHW=WTHW+(W2C-W1C)
      WTHU=WTHU+(W2CU-W1CU)
      WTHF=WTHF+(W2F-W1F)
      WRITE(1,100)M,N,V1,V2,D
100   FORMAT(5X,*M=*,I3,*N=*,I3,*V1=*,F8.4,*V2=*,F8.4
```

```
     1,*D=*,F8.4,/)
      WRITE(1,102)W1C,W2C,W1CU,W2CU,W1F,W2F
102   FORMAT(/,5X,*W1C=*,F10.4,*W2C=*,F10.4,*W1CU=*,
     1F10.4,*W2CU=*,F10.4,*W1F=*,F10.4,*W2F=*,F10.4,/)
      L=L+1
      IF(L.LE.99)GO TO 15
      WTHW=WTHW/100.0
      WTHU =WTHU/100.0
      WTHF = WTHF/100.0
      WRITE(1,103)
103   FORMAT(/,5X,*AVERAGE WIDTHS=*)
      WRITE(1,104) WTHW,WTHU,WTHF
104   FORMAT(/,5X,*WTHW=*,F10.4,*WTHU=*,F10.4,
     1  *WTHF=*,F10.4/)
      WRITE(1,105)
105   FORMAT(/,5X,*COVERAGE FREQUENCY=*)
      WRITE(1,106)IFW,IFU,IFF
106   FORMAT(/,5X,*1FW=*,I4,*IFU=*,I4,*IFF=*,I4)
12    CONTINUE
11    CONTINUE
10    CONTINUE
      STOP
      END
        SUBROUTINE HPDW(M,N,V1,V2,D,ALPHA,CMIN,CMAX,W1C,W2C)
C     COMPUTES HPD INTERVAL BASED ON W
      COMMON /BL/CONST
        EXTERNAL F,HPDC
      CONST = DCADRE(F,0.0E0,1.0E0,1.0E-25,1.0E-10,ERR,IER)
      ITMAX1=100
      EPS1=1.0E-10
      XL1 = CMIN
      XR1 = CMAX
       CALL ZEROHPD(HPDC,XL1,XR1,EPS1,ITMAX1,CSOL)
      RETURN
      END
      SUBROUTINE SOLMOD(W)
C     COMPUTTES THE MODE OF P(W/Y)
      COMMON /BLI/M,N,V1,V2,D
      EXTERNAL H
      EPS= 1.0E-10
      NSIG =10
      XL = 1.0E-20
      XR = (V1/V2)+20.0
      ITMAX =100
      CALL ZFALSE(H,EPS,NSIG,XL,XR,W,ITMAX,IER)
      RETURN
      END
      FUNCTION H(W)
      COMMON /BLI/M,N,V1,V2,D
      T1=(N/2.0-1.0)/W
      T2 = N/(2.0*(M+N*W))
      T3 = M+N-1.0
      T4 = (N-1.0)*V2+N*M*M*D*D  /((M+N*W)**2.0)
      T6 = 2.0*((M-1.0)*V1+(N-1.0)*V2*W)
      T7 = M*N*W*D*D/(M+N*W)
      FP = T3*T4
```

```
      FP = FP/(T6+2.0*T7)
      H = T1-T2-FP
      RETURN
      END
      FUNCTION   HPDC(C)
      COMMON /BL/CONST/BL1/ALPHA/BL3/PRC/UBL/W1C,W2C/BLI/M,N,
     1V1,V2,D
      EXTERNAL F1,F2
C     THIS FUNCTION FINDS THE VALUES W1C,W2C SUCH THAT
C     THE INTERVAL (W1C,W2C) HAVE THE SAME POSTERIOR
C     ORDINATES FOR A GIVEN C, THE VALUE OF ORDINATE
C      BEING THE PRODUCT OF C AND THE POSTERIOR ORDINATE
C      AT THE MODE.
C       AND THEN GIVES THE PROBABILITY CONTENT OF THIS
C       INTERVAL
      CALL SOLMOD(W)
      WRITE(1,700) W
700   FORMAT(2X,F13.6)
      X = W / (1.+W)
      XL=1.0E-20
      XR=1.0-(1.0E-20)
      PRX = F1(X)
      PRW=PRX*(1.-X)**2.0
      PRC = C*PRW
      ITMAX = 100
      XM=X
      CALL ZFALSE(F2,1.0E-10,10,XL,XM ,X1C,ITMAX,IER)
        ITMAX = 100
      XM=X
      CALL ZFALSE(F2,1.0E-10,10,XM,XR ,X2C,ITMAX,IER)
      U1C = ALOG(X1C/(1.-X1C))
      U2C = ALOG(X2C/(1.-X2C))
      W1C = EXP(U1C)
      W2C = EXP(U2C)
      PROBC = DCADRE(F1,X1C,X2C,1.0E-100,1.0E-50,ERR,IER)
      HPDC = PROBC-(1.-ALPHA)
      RETURN
      END
      FUNCTION F2(X)
      COMMON /BL3/PRC
      F2=F1(X)*(1.-X)**2.0-PRC
      RETURN
      END
      FUNCTION F(X)
      COMMON /BLI/M,N,V1,V2,D
      IF(X .EQ. 0.0E0 .OR. X .EQ. 1.0E0) GO TO 1
      W=X/(1-X)
      N1=M-1
      N2=N-1
      T1=(N1*V1+N2*W*V2)
      T2=M*N*W*D*D/(M+N*W)
      FP=(M+N*W)
      FP=W**((N/2.0)-1.0)*FP**(-0.5)*((1.0-X)**(-2.0))
      F = FP/(T1+T2)**((M+N-1.0)/2.0)
      GO TO 2
1     F = 0.0E0
2     RETURN
```

```
      END
      FUNCTION F1(X)
      COMMON /BL/CONST
      F1 = F(X)/CONST
      RETURN
      END
      SUBROUTINE ZEROHPD(HPD,XMIN,XMAX,ERROR,ITM,XSOL)
C     FINDS THE SOLUTION SUCH HPD IS ZERO
C     SOLUTION XSOL LIES BETWEEN XMIN AND XMAX
C     ITM = MAX NO OF ITERATIONS
C     ERROR = ERROR BOUND
      FMIN = HPD(XMIN)
      FMAX = HPD(XMAX)
      ITMX = 1
      IF( (FMIN.GT.0.0 .AND. FMAX.GT.0.0) .OR.
     1(FMIN.LT.0.0 .AND. FMAX.LT.0.0)) GO TO 2
4     X = (XMIN+XMAX)/2.0
      F = HPD(X)
      IF(ABS(F).LE.ERROR) GO TO 6
      ITMX = ITMX + 1
      IF( ITMX.GT.ITM) GO TO 5
      IF(F.GT.0.0) GO TO 3
      XMAX = X
      GO TO 4
3     XMIN = X
      GO TO 4
5     PRINT*,"NO. OF ITERATIONS HAVE REACHED MAXIMUM. VALUE OF
     1 HPD=",F
      GO TO 1
2     PRINT*,"SOLUTION DOES NOT LIE IN THE GIVEN INTERVAL"
      GO TO 1
6      XSOL = X
1     RETURN
      END
        SUBROUTINE HPDW1(M,N,V1,V2,D,ALPHA,CMIN,CMAX,W1CU,W2CU)
C     COMPUTES HPD INTERVAL BASED ON U
      COMMON /BL/CONST
      EXTERNAL HPDC1
      ITMAX2=100
      EPS2=1.0E-10
      XL2=CMIN
      XR2 = CMAX
      CALL ZEROHPD(HPDC1,XL2,XR2,EPS2,ITMAX2,CSOL1)
      RETURN
      END
      SUBROUTINE SOLMOD1(W)
C       COMPUTES THE MODE OF P(LOGW/Y)
      COMMON /BLI/M,N,V1,V2,D
      EXTERNAL H1
      EPS= 1.0E-10
      NSIG =10
      XL = 1.0E-10
      XR = (V1/V2)+20.0
      ITMAX =100
      CALL ZFALSE(H1,EPS,NSIG,XL,XR,W,ITMAX,IER)
      RETURN
```

```
      END
      FUNCTION H1(W)
      COMMON /BLI/M,N,V1,V2,D
      T1=N/(2*W)
      T2 = N/(2.0*(M+N*W))
      T3 = M+N-1.0
      T4 = (N-1.0)*V2+N*M*M*D*D  /((M+N*W)**2.0)
      T6 = 2.0*((M-1.0)*V1+(N-1.0)*V2*W)
      T7 = M*N*W*D*D/(M+N*W)
      FP = T3*T4
      FP = FP/(T6+2.0*T7)
      H1 = T1-T2-FP
      RETURN
      END
      FUNCTION   HPDC1(C)
      COMMON /BL/CONST/BL1/ALPHA/UBL3/PRC1/UBL1/W1CU,W2CU
     1/BLI/M,N,V1,V2,D
      EXTERNAL F1,F2U
C     THIS FUNCTION FINDS THE VALUES W1C,W2C SUCH THAT
C     THE INTERVAL (W1C,W2C) HAVE THE SAME POSTERIOR
C     ORDINATES FOR A GIVEN C, THE VALUE OF ORDINATE OF LOG(W)
C       BEING THE PRODUCT OF C AND THE POSTERIOR ORDINATE
C       AT THE MODE.
C        AND THEN GIVES THE PROBABILITY CONTENT OF THIS
C        INTERVAL
      CALL SOLMOD1(W)
      X = W / (1.+W)
      XL=1.0E-10
      XR=1.0-(1.0E-10)
      PRX = F1(X)
      PRW=PRX*(1.-X)**2.0
      PRU=PRW*W
      PRC1 = C*PRU
      ITMAX = 100
      XM = X
      CALL ZFALSE(F2U,1.0E-10,6,XL,XM, X1C,ITMAX,IER)
      XM = X
        ITMAX = 100
      CALL ZFALSE(F2U,1.0E-10,6,XM,XR,X2C,ITMAX,IER)
      U1C = ALOG(X1C/(1.-X1C))
      U2C = ALOG(X2C/(1.-X2C))
      W1CU = EXP(U1C)
      W2CU = EXP(U2C)
      PROBC = DCADRE(F1,X1C,X2C,1.0E-100,1.0E-50,ERR,IER)
      HPDC1 = PROBC-(1.-ALPHA)
      RETURN
      END
       FUNCTION F2U(X)
       COMMON /UBL3/PRC1
       F2U = F1(X)*X*(1.-X)-PRC1
       RETURN
       END
```

APPENDIX A3.1

```
      PROGRAM HPDINT(OUTPUT,TAPE6)
C     THIS PROGRAM COMPUTES THE HPD-INTERVALS BASED
C     ON  W AND U FOR BOOT AND DEWITT'S DATA
        COMMON /UBL/W1C,W2C/BL1/ALPHA/UBL1/W1CU,W2CU
      DATA ALPHA/.05/,CMIN/.0001/,CMAX/.9999/
      WRITE(1,101)ALPHA
101   FORMAT(5X,*ALPHA=*,F10.4)
      CALL HPDW(ALPHA,CMIN,CMAX,W1C,W2C)
      CALL HPDW1(ALPHA,CMIN,CMAX,W1CU,W2CU)
      WRITE(1,102)W1C,W2C,W1CU,W2CU
102   FORMAT(/,5X,*W1C=*,F10.4,*W2C=*,F10.4,*W1CU=*,
     1F10.4,*W2CU=*,F10.4,/)
      STOP
      END
        SUBROUTINE HPDW(ALPHA,CMIN,CMAX,W1C,W2C)
C     COMPUTES HPD INTERVAL BASED ON W
      COMMON /BL/CONST
        EXTERNAL F,HPDC
      CONST = DCADRE(F,0.0E0,1.0E0,1.0E-25,1.0E-10,ERR,IER)
      ITMAX1=100
      EPS1=1.0E-10
      XL1 = CMIN
      XR1 = CMAX
        CALL ZEROHPD(HPDC,XL1,XR1,EPS1,ITMAX1,CSOL)
      RETURN
      END
      SUBROUTINE SOLMOD(W)
C     COMPUTTES THE MODE OF P(W/Y)
      EXTERNAL H
      EPS= 1.0E-10
      NSIG =10
      XL  = 1.0E-20
      XR  = (V1/V2)+20.0
      ITMAX =100
      CALL ZFALSE(H,EPS,NSIG,XL,XR,W,ITMAX,IER)
      RETURN
      END
      FUNCTION H(W)
      T1= 17.0/W
      T2 = .02703/(1.0+.02703*W)
      T3 = .30513/(1.0+.30513*W)
      T4=17*(777.0+104.0*W)
      SSGW=(T4+W*(T2*5188.3209+T3*534.5344))/36.0
      T4=17.0*104.0
      T5=.02703*5188.3209/(1.0+.02703*W)**2.0
      T6=.30513*534.5344/(1.0+.30513*W)**2.0
      H = T1-T2-T3-(T4+T5+T6)/SSGW
      RETURN
      END
      FUNCTION   HPDC(C)
      COMMON /BL/CONST/BL1/ALPHA/BL3/PRC/UBL/W1C,W2C
      EXTERNAL F1,F2
C     THIS FUNCTION FINDS THE VALUES W1C,W2C SUCH THAT
C     THE INTERVAL (W1C,W2C) HAVE THE SAME POSTERIOR
C     ORDINATES FOR A GIVEN C, THE VALUE OF ORDINATE
C       BEING THE PRODUCT OF C AND THE POSTERIOR ORDINATE
```

```
C        AT THE MODE.
C        AND THEN GIVES THE PROBABILITY CONTENT OF THIS
C        INTERVAL
      CALL SOLMOD(W)
      X = W / (1.+W)
      XL=1.0E-20
      XR=1.0-(1.0E-20)
      PRX = F1(X)
      PRW=PRX*(1.-X)**2.0
      PRC = C*PRW
      ITMAX = 100
      XM=X
      CALL ZFALSE(F2,1.0E-10,10,XL,XM ,X1C,ITMAX,IER)
        ITMAX = 100
      XM=X
      CALL ZFALSE(F2,1.0E-10,10,XM,XR ,X2C,ITMAX,IER)
      U1C = ALOG(X1C/(1.-X1C))
      U2C = ALOG(X2C/(1.-X2C))
      W1C = EXP(U1C)
      W2C = EXP(U2C)
      PROBC = DCADRE(F1,X1C,X2C,1.0E-100,1.0E-50,ERR,IER)
      HPDC = PROBC-(1.-ALPHA)
      RETURN
      END
      FUNCTION F2(X)
      COMMON /BL3/PRC
      F2=F1(X)*(1.-X)**2.0-PRC
      RETURN
      END
      FUNCTION F(X)
      IF(X .EQ. 0.0E0 .OR. X .EQ. 1.0E0) GO TO 1
      W=X/(1-X)
      T1=W*.02703/(1.0+.02703*W)
      T2=W*.30513/(1.0+.30513*W)
      T3=17*(777.0+104.0*W)
      SSGW=(T3+T1*5188.3209+T2*534.5344)/36.0
      FP=(1.0+.02703*W)*(1.0+.30513*W)
      F=W**8.5*((1.0-X)**(-2.0))*SSGW**(-18.0)*FP**(-0.5)
      GO TO 2
1     F = 0.0E0
2     RETURN
      END
      FUNCTION F1(X)
      COMMON /BL/CONST
      F1 = F(X)/CONST
      RETURN
      END
      SUBROUTINE ZEROHPD(HPD,XMIN,XMAX,ERROR,ITM,XSOL)
C     FINDS THE SOLUTION SUCH HPD IS ZERO
C     SOLUTION XSOL LIES BETWEEN XMIN AND XMAX
C     ITM = MAX NO OF ITERATIONS
C     ERROR = ERROR BOUND
      FMIN = HPD(XMIN)
      FMAX = HPD(XMAX)
      ITMX = 1
      IF( (FMIN.GT.0.0 .AND. FMAX.GT.0.0) .OR.
```

```
     1(FMIN.LT.0.0 .AND. FMAX.LT.0.0)) GO TO 2
4     X = (XMIN+XMAX)/2.0
      F = HPD(X)
      IF(ABS(F).LE.ERROR) GO TO 6
      ITMX = ITMX + 1
      IF( ITMX.GT.ITM) GO TO 5
      IF(F.GT.0.0) GO TO 3
      XMAX = X
      GO TO 4
3     XMIN = X
      GO TO 4
5     PRINT*,"NO. OF ITERATIONS HAVE REACHED MAXIMUM, VALUE OF
     1 HPD=",F
      GO TO 1
2     PRINT*,"SOLUTION DOES NOT LIE IN THE GIVEN INTERVAL"
      GO TO 1
6      XSOL = X
1     RETURN
      END
        SUBROUTINE HPDW1(ALPHA,CMIN,CMAX,W1CU,W2CU)
C     COMPUTES HPD INTERVAL BASED ON U.
      COMMON /BL/CONST
      EXTERNAL HPDC1
      ITMAX2=100
      EPS2=1.0E-10
      XL2=CMIN
      XR2 = CMAX
      CALL ZEROHPD(HPDC1,XL2,XR2,EPS2,ITMAX2,CSOL1)
      RETURN
      END
      SUBROUTINE SOLMOD1(W)
C       COMPUTES THE MODE OF P(LOGW/Y)
      COMMON /BLI/M,N,V1,V2,D
      EXTERNAL H1
      EPS= 1.0E-10
      NSIG =10
      XL = 1.0E-10
      XR = (V1/V2)+20.0
      ITMAX =100
      CALL ZFALSE(H1,EPS,NSIG,XL,XR,W,ITMAX,IER)
      RETURN
      END
      FUNCTION H1(W)
      T1=19/W
      T2 = .02703/(1.0+.02703*W)
      T3 =.30513/(1.0+.30513*W)
      T4 = .02703*5188.3209/(1.0+.02703*W)**2.0
      T5=.30513*534.5344/(1.0+.30513*W)**2.0
      T6=17*(777.0+104.0*W)
      SSGW=(T6+W*(T2*5188.3209+T3*534.5344))/36.0
      T7=17*104.0
      H1 = T1-T2-T3-(T7+T4+T5)/SSGW
      RETURN
      END
      FUNCTION   HPDC1(C)
      COMMON /BL/CONST/BL1/ALPHA/UBL3/PRC1/UBL1/W1CU,W2CU
```

```
      EXTERNAL F1,F2U
C     THIS FUNCTION FINDS THE VALUES W1C,W2C SUCH THAT
C     THE INTERVAL (W1C,W2C) HAVE THE SAME POSTERIOR
C     ORDINATES FOR A GIVEN C, THE VALUE OF ORDINATE OF LOG(W)
C      BEING THE PRODUCT OF C AND THE POSTERIOR ORDINATE
C      AT THE MODE.
C       AND THEN GIVES THE PROBABILITY CONTENT OF THIS
C       INTERVAL
      CALL SOLMOD1(W)
      X = W / (1.+W)
      XL=1.0E-10
      XR=1.0-(1.0E-10)
      PRX = F1(X)
      PRW=PRX*(1.-X)**2.0
      PRU=PRW*W
      PRC1 = C*PRU
      ITMAX = 100
      XM = X
      CALL ZFALSE(F2U,1.0E-10,6,XL,XM, X1C,ITMAX,IER)
      XM = X
        ITMAX = 100
      CALL ZFALSE(F2U,1.0E-10,6,XM,XR,X2C,ITMAX,IER)
      U1C = ALOG(X1C/(1.-X1C))
      U2C = ALOG(X2C/(1.-X2C))
      W1CU = EXP(U1C)
      W2CU = EXP(U2C)
      PROBC = DCADRE(F1,X1C,X2C,1.0E-100,1.0E-50,ERR,IER)
      HPDC1 = PROBC-(1.-ALPHA)
      RETURN
      END
       FUNCTION F2U(X)
       COMMON /UBL3/PRC1
       F2U = F1(X)*X*(1.-X)-PRC1
       RETURN
       END
```